

# LOCALIZATION OF $\mathfrak{u}$ -MODULES. III. TENSOR CATEGORIES ARISING FROM CONFIGURATION SPACES.

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## 1. INTRODUCTION

1.1. This article is a sequel to [FS].

In Chapter 1 we associate with every Cartan matrix of finite type and a non-zero complex number  $\zeta$  an abelian artinian category  $\mathcal{FS}$ . We call its objects *finite factorizable sheaves*. They are certain infinite collections of perverse sheaves on configuration spaces, subject to a compatibility ("factorization") and finiteness conditions.

In Chapter 2 the tensor structure on  $\mathcal{FS}$  is defined using functors of nearby cycles. It makes  $\mathcal{FS}$  a braided tensor category.

In Chapter 3 we define, using vanishing cycles functors, an exact tensor functor

$$\Phi : \mathcal{FS} \longrightarrow \mathcal{C}$$

to the category  $\mathcal{C}$  connected with the corresponding quantum group, cf. [AJS], 1.3 and [FS] II, 11.3, 12.2.

In Chapter 4 we prove

1.2. **Theorem.**  *$\Phi$  is an equivalence of categories.*

One has to distinguish two cases.

(i)  $\zeta$  is not root of unity. In this case it is wellknown that  $\mathcal{C}$  is semisimple. This case is easier to treat; 1.2 is Theorem 18.4.

(ii)  $\zeta$  is a root of unity. This is of course the most interesting case; 1.2 is Theorem 17.1.

$\Phi$  may be regarded as a way of localizing  $\mathfrak{u}$ -modules from category  $\mathcal{C}$  to the origin of the affine line  $\mathbb{A}^1$ . More generally, in order to construct tensor structure on  $\mathcal{FS}$ , we define for each finite set  $K$  certain categories  ${}^K\mathcal{FS}$  along with the functor  $g_K : \mathcal{FS}^K \longrightarrow {}^K\mathcal{FS}$  (see the Sections 9, 10) which may be regarded as a way of localizing  $K$ -tuples of  $\mathfrak{u}$ -modules to  $K$ -tuples of points of the affine line. In a subsequent paper we plan to show

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*Date:* Second Edition

May 1995

q-alg/9503013.

The second author was supported in part by NSF grant DMS-9202280.

how to localize  $\mathfrak{u}$ -modules to the points of an arbitrary smooth curve. For example, the case of a projective line is already quite interesting, and is connected with "semiinfinite" cohomology of quantum groups.

We must warn the reader that the proofs of some technical topological facts are only sketched in this paper. The full details will appear later on.

1.3. The construction of the space  $\mathcal{A}$  in Section 2 is inspired by the idea of "semiinfinite space of divisors on a curve" one of us learnt from A.Beilinson back in 1990. The construction of the braiding local system  $\mathcal{I}$  in Section 3 is very close in spirit to P.Deligne's letter [D1]. In terms of this letter, all the local systems  $\mathcal{I}_\mu^\alpha$  arise from the semisimple braided tensor category freely generated by the irreducibles  $i \in I$  with the square of  $R$ -matrix:  $i \otimes j \longrightarrow j \otimes i \longrightarrow i \otimes j$  given by the scalar matrix  $\zeta^{i \cdot j}$ .

We are very grateful to B.Feigin for the numerous inspiring discussions, and to P.Deligne and L.Positselsky for the useful comments concerning the definition of morphisms in  ${}^K\mathcal{FS}$  in 9.6.

1.4. **Notations.** We will use all the notations of [FS]. References to *loc.cit.* will look like Z.1.1 where Z=I or II.

During the whole paper we fix a Cartan datum  $(I, \cdot)$  of finite type and denote by  $(Y = \mathbb{Z}[I]; \mathbb{X} = \text{Hom}(\mathbb{Y}, \mathbb{Z}); I \hookrightarrow \mathbb{Y}, \beth \mapsto \beth; I \hookrightarrow \mathbb{X}, \beth \mapsto \beth')$  the simply connected root datum associated with  $(I, \cdot)$ , [L], 2.2.2. Given  $\alpha = \sum a_i i \in Y$ , we will denote by  $\alpha'$  the element  $\sum a_i i' \in X$ . This defines an embedding

$$Y \hookrightarrow X \tag{1}$$

We will use the notation  $d_i := \frac{i \cdot i}{2}$ . We have  $\langle j', d_i i \rangle = i \cdot j$ . We will denote by  $A$  the  $I \times I$ -matrix  $(\langle i, j' \rangle)$ . We will denote by  $\lambda, \mu \mapsto \lambda \cdot \mu$  a unique  $\mathbb{Z}[\frac{\mathbb{K}}{\det A}]$ -valued scalar product on  $X$  such that (1) respects scalar products. We have

$$\lambda \cdot i' = \langle \lambda, d_i i \rangle \tag{2}$$

for each  $\lambda \in X$ ,  $i \in I$ .

We fix a non-zero complex number  $\zeta'$  and suppose that our ground field  $B$  contains  $\zeta'$ . We set  $\zeta := (\zeta')^{\det A}$ ; for  $a = \frac{c}{\det A}$ ,  $c \in \mathbb{Z}$ , we will use the notation  $\zeta^a := (\zeta')^c$ .

We will use the following partial orders on  $X$  and  $Y$ . For  $\alpha = \sum a_i i$ ,  $\beta = \sum b_i i \in Y$  we write  $\alpha \leq \beta$  if  $a_i \leq b_i$  for all  $i$ . For  $\lambda, \mu \in X$  we write  $\lambda \geq \mu$  if  $\lambda - \mu = \alpha'$  for some  $\alpha \in Y$ ,  $\alpha \geq 0$ .

1.5. If  $X_1, X_2$  are topological spaces,  $\mathcal{K}_\gamma \in \mathcal{D}(\mathcal{X}_\gamma)$ ,  $\gamma = \infty, \in$ , we will use the notation  $\mathcal{K}_\infty \boxtimes \mathcal{K}_\in$  for  $p_1^* \mathcal{K}_\infty \otimes \bigvee_{\in}^* \mathcal{K}_\in$  (where  $p_i : X_1 \times X_2 \longrightarrow X_i$  are projections). If  $J$  is a finite set,  $|J|$  will denote its cardinality.

For a constructible complex  $\mathcal{K}$ ,  $SS(\mathcal{K})$  will denote the singular support of  $\mathcal{K}$  (micro-support in the terminology of [KS], cf. *loc.cit.*, ch. V).

## CHAPTER 1. Category $\mathcal{FS}$

### 2. SPACE $\mathcal{A}$

2.1. We will denote by  $\mathbb{A}^\#$  the complex affine line with a fixed coordinate  $t$ . Given real  $c, c', 0 \leq c < c'$ , we will use the notations  $D(c) = \{t \in \mathbb{A}^\# \mid |\approx| < c\}$ ;  $\bar{\mathbb{D}}() = \{\approx \in \mathbb{A}^\# \mid |\approx| \leq c\}$ ;  $\mathbb{A}_>^\# := \{\approx \in \mathbb{A}^\# \mid |\approx| > c\}$ ;  $D(c, c') = \mathbb{A}_>^\# \cap \bar{\mathbb{D}}(c')$ .

Recall that we have introduced in II.6.12 configuration spaces  $\mathcal{A}_\alpha$  for  $\alpha \in \mathbb{N}[\mathbb{I}]$ . If  $\alpha = \sum a_i i$ , the space  $\mathcal{A}_\alpha$  parametrizes configurations of  $I$ -colored points  $\mathbf{t} = (t_j)$  on  $\mathbb{A}^\#$ , such that there are precisely  $a_i$  points of color  $i$ .

2.2. Let us introduce some open subspaces of  $\mathcal{A}_\alpha$ . Given a sequence

$$\vec{\alpha} = (\alpha_1, \dots, \alpha_p) \in \mathbb{N}[\mathbb{I}]^p \quad (3)$$

and a sequence of real numbers

$$\vec{d} = (d_1, \dots, d_{p-1}) \quad (4)$$

such that  $0 < d_{p-1} < d_{p-2} < \dots < d_1$ ,  $p \geq 2$ , we define an open subspace

$$\mathcal{A}^{\vec{\alpha}}(\vec{d}) \subset \mathcal{A}_\alpha \quad (5)$$

which parametrizes configurations  $\mathbf{t}$  such that  $\alpha_p$  of points  $t_j$  lie inside the disk  $D(d_{p-1})$ , for  $2 \leq i \leq p-1$ ,  $\alpha_i$  of points lie inside the annulus  $D(d_{i-1}, d_i)$ , and  $\alpha_1$  of points lie inside the ring  $\mathcal{A}_{>[\infty}^\infty$ .

For  $p = 1$ , we set  $\mathcal{A}^\alpha(\emptyset) := \mathcal{A}_\alpha$ .

By definition, a configuration space of empty collections of points consists of one point. For example, so is  $\mathcal{A}'(\emptyset)$ .

2.2.1. *Cutting.* Given  $i \in [p-1]$  define subsequences

$$\vec{d}_{\leq i} = (d_1, \dots, d_i); \quad \vec{d}_{\geq i} = (d_i, \dots, d_{p-1}) \quad (6)$$

and

$$\vec{\alpha}_{\leq i} = (\alpha_1, \dots, \alpha_i, 0); \quad \vec{\alpha}_{\geq i} = (0, \alpha_{i+1}, \dots, \alpha_p) \quad (7)$$

We have obvious *cutting isomorphisms*

$$c_i : \mathcal{A}^{\vec{\alpha}}(\vec{d}) \xrightarrow{\sim} \mathcal{A}^{\vec{\alpha}_{\leq i}}(\vec{d}_{\leq i}) \times \mathcal{A}^{\vec{\alpha}_{\geq i}}(\vec{d}_{\geq i}) \quad (8)$$

satisfying the following compatibility:

for  $i < j$  the square

$$\begin{array}{ccc}
& \mathcal{A}^{\vec{\alpha}}(\vec{\Gamma}) & \\
c_j \swarrow & & \searrow c_i \\
\mathcal{A}^{\vec{\alpha}_{\leq i}}(\vec{\Gamma}_{\leq i}) \times \mathcal{A}^{\vec{\alpha}_{\geq i}}(\vec{\Gamma}_{\geq i}) & & \mathcal{A}^{\vec{\alpha}_{\leq i}}(\vec{\Gamma}_{\leq i}) \times \mathcal{A}^{\vec{\alpha}_{\geq i}}(\vec{\Gamma}_{\geq i}) \\
c_i \times \text{id} \searrow & & \swarrow \text{id} \times c_j \\
\mathcal{A}^{\vec{\alpha}_{\leq i}}(\vec{\Gamma}_{\leq i}) \times \mathcal{A}^{\vec{\alpha}_{\geq i}; \leq i}(\vec{\Gamma}_{\geq i}; \leq i) \times \mathcal{A}^{\vec{\alpha}_{\geq i}}(\vec{\Gamma}_{\geq i}) & & 
\end{array}$$

commutes.

2.2.2. *Dropping.* For  $i$  as above, let  $\partial_i \vec{d}$  denote the subsequence of  $\vec{d}$  obtained by dropping  $d_i$ , and set

$$\partial_i \vec{\alpha} = (\alpha_1, \dots, \alpha_{i-1}, \alpha_i + \alpha_{i+1}, \alpha_{i+2}, \dots, \alpha_p) \quad (9)$$

We have obvious open embeddings

$$\mathcal{A}^{\partial_i \vec{\alpha}}(\partial_i \vec{\Gamma}) \hookrightarrow \mathcal{A}^{\vec{\alpha}}(\vec{\Gamma}) \quad (10)$$

2.3. Let us define  $\mathcal{A}_\mu^{\vec{\alpha}}(\vec{\Gamma})$  as the space  $\mathcal{A}^{\vec{\alpha}}(\vec{\Gamma})$  equipped with an additional index  $\mu \in X$ . One should understand  $\mu$  as a weight assigned to the origin in  $\mathbb{A}^{\mathbb{K}}$ . We will abbreviate the notation  $\mathcal{A}_\mu^\alpha(\emptyset)$  to  $\mathcal{A}_\mu^\alpha$ .

Given a triple  $(\vec{\alpha}, \vec{d}, \mu)$  as above, let us define its  $i$ -cutting — two triples  $(\vec{\alpha}_{\leq i}, \vec{d}_{\leq i}, \mu_{\leq i})$  and  $(\vec{\alpha}_{\geq i}, \vec{d}_{\geq i}, \mu)$ , where

$$\vec{\mu}_{\leq i} = \vec{\mu} - \left( \sum_{j=i+1}^p \alpha_j \right)'.$$

We will also consider triples  $(\partial_i \vec{\alpha}, \partial_i \vec{d}, \partial_i \mu)$  where  $\partial_i \mu = \mu$  if  $i < p-1$ , and  $\partial_{p-1} \mu = \mu - \alpha'_p$ .

The cutting isomorphisms (8) induce isomorphisms

$$\mathcal{A}_\mu^{\vec{\alpha}}(\vec{\Gamma}) \xrightarrow{\sim} \mathcal{A}_{\mu_{\leq i}}^{\vec{\alpha}_{\leq i}}(\vec{\Gamma}_{\leq i}) \times \mathcal{A}_\mu^{\vec{\alpha}_{\geq i}}(\vec{\Gamma}_{\geq i}) \quad (11)$$

2.4. For each  $\mu \in X$ ,  $\alpha = \sum a_i i, \beta = \sum b_i i \in \mathbb{N}[\mathbb{I}]$ ,

$$\sigma = \sigma_\mu^{\alpha, \beta} : \mathcal{A}_\mu^\alpha \hookrightarrow \mathcal{A}_{\mu+\beta}^{\alpha+\beta} \quad (12)$$

will denote a closed embedding which adds  $b_i$  points of color  $i$  equal to 0.

For  $d > 0$ ,  $\mathcal{A}_{\mu+\beta}^{(\alpha, \beta)}(\Gamma)$  is an open neighbourhood of  $\sigma(\mathcal{A}_\mu^\alpha)$  in  $\mathcal{A}_{\mu+\beta}^{\beta+\alpha}$ .

2.5. By definition,  $\mathcal{A}$  is a collection of all spaces  $\mathcal{A}_\mu^{\vec{\alpha}}(\vec{\Gamma})$  as above, together with the cutting isomorphisms (11) and the closed embeddings (12).

2.6. Given a coset  $c \in X/Y$  (where we regard  $Y$  as embedded into  $X$  by means of a map  $i \mapsto i'$ ), we define  $\mathcal{A}_\downarrow$  as a subset of  $\mathcal{A}$  consisting of  $\mathcal{A}_\mu^\alpha$  such that  $\mu \in c$ . Note that the closed embeddings  $\sigma$ , as well as cutting isomorphisms act inside  $\mathcal{A}_\downarrow$ . This subset will be called a *connected component* of  $\mathcal{A}$ . The set of connected components will be denoted  $\pi_0(\mathcal{A})$ . Thus, we have canonically  $\pi_0(\mathcal{A}) \cong \mathcal{X}/\mathcal{Y}$ .

2.7. We will be interested in two stratifications of spaces  $\mathcal{A}_\mu^\alpha$ . We will denote by  $\dot{\mathcal{A}}_\mu^\alpha \subset \mathcal{A}_\mu^\alpha$  the complement

$$\mathcal{A}_\mu^\alpha - \bigcup_{\beta < \alpha} \sigma(\mathcal{A}_{\mu-\beta'+\alpha'}^\beta).$$

We define a *toric stratification* of  $\mathcal{A}_\mu^\alpha$  as

$$\mathcal{A}_\mu^\alpha = \coprod \sigma(\dot{\mathcal{A}}_{\mu-\beta'+\alpha'}^\beta).$$

Another stratification of  $\mathcal{A}_\mu^\alpha$  is the *principal stratification* defined in II.7.14. Its open stratum will be denoted by  $\overset{\circ}{\mathcal{A}}_\mu^\alpha \subset \mathcal{A}_\mu^\alpha$ . Unless specified otherwise, we will denote the principal stratification on spaces  $\mathcal{A}_\mu^\alpha$ , as well as the induced stratifications on its subspaces, by  $\mathcal{S}$ .

The sign  $\circ$  (resp.,  $\bullet$ ) over a subspace of  $\mathcal{A}_\mu^\alpha$  will denote the intersection of this subspace with  $\overset{\circ}{\mathcal{A}}_\mu^\alpha$  (resp., with  $\dot{\mathcal{A}}_\mu^\alpha$ ).

### 3. BRAIDING LOCAL SYSTEM $\mathcal{I}$

3.1. **Local systems  $\mathcal{I}_\mu^\alpha$ .** Let us recall some definitions from II. Let  $\alpha = \sum_i a_i i \in \mathbb{N}[\mathbb{I}]$  be given. Following II.6.12, let us choose an *unfolding* of  $\alpha$ , i.e. a set  $J$  together with a map  $\pi : J \longrightarrow I$  such that  $|\pi^{-1}(i)| = a_i$  for all  $i$ . We define the group  $\Sigma_\pi := \{\sigma \in \text{Aut}(J) \mid \sigma \circ \pi = \pi\}$ .

We define  ${}^\pi\mathbb{A}$  as an affine space with coordinates  $t_j$ ,  $j \in J$ ; it is equipped with the principal stratification defined by hyperplanes  $t_j = 0$  and  $t_i = t_j$ , cf. II.7.1. The group  $\Sigma_\pi$  acts on  ${}^\pi\mathbb{A}$  by permutations of coordinates, respecting the stratification. By definition,  $\mathcal{A}_\alpha = {}^\pi\mathbb{A}/\Sigma_\pi$ . We will denote by the same letter  $\pi$  the canonical projection  ${}^\pi\mathbb{A} \longrightarrow \mathcal{A}_\alpha$ .

If  ${}^\pi\overset{\circ}{\mathbb{A}} \subset {}^\pi\mathbb{A}$  denotes the open stratum of the principal stratification,  $\pi({}^\pi\overset{\circ}{\mathbb{A}}) = \overset{\circ}{\mathcal{A}}_\alpha$ , and the restriction of  $\pi$  to  ${}^\pi\overset{\circ}{\mathbb{A}}$  is unramified covering.

Suppose a weight  $\mu \in X$  is given. Let us define a one dimensional local system  ${}^\pi\mathcal{I}_\mu$  over  ${}^\pi\overset{\circ}{\mathbb{A}}$  by the procedure II.8.1. Its fiber over each positive chamber  $C \in \pi_0({}^\pi\overset{\circ}{\mathbb{A}}_\mathbb{R})$  is identified with  $B$ ; and monodromies along standard paths shown on II, Fig. 5 (a), (b) are given by the formulas

$${}^CT_{ij} = \zeta^{-\pi(i) \cdot \pi(j)}, \quad {}^CT_{i0} = \zeta^{2\mu \cdot \pi(i)} \quad (13)$$

respectively (cf. (2)). (Note that, by technical reasons, this definition differs by the sign from that of II.8.2 and II.12.6).

We have a canonical  $\Sigma_\pi$ -equivariant structure on  ${}^\pi\mathcal{I}_\mu$ , i.e. a compatible system of isomorphisms

$$i_\sigma : {}^\pi\mathcal{I}_\mu \xrightarrow{\sim} \sigma^* {}^\pi\mathcal{I}_\mu, \quad \sigma \in \Sigma_\pi, \quad (14)$$

defined uniquely by the condition that

$$(i_\sigma)_C = \text{id}_B : ({}^\pi\mathcal{I}_\mu)_C = \mathcal{B} \xrightarrow{\sim} (\sigma^* {}^\pi\mathcal{I}_\mu)_{\sigma C} = \mathcal{B}$$

for all (or for some) chamber  $C$ . As a consequence, the group  $\Sigma_\pi$  acts on the local system  $\pi_* {}^\pi\mathcal{I}_\mu$ .

Let  $\text{sgn} : \Sigma_\pi \rightarrow \{\pm 1\}$  denote the sign character. We define a one-dimensional local system  $\mathcal{I}_\mu^\alpha$  over  $\overset{\circ}{\mathcal{A}}_\mu^\alpha = \overset{\circ}{\mathcal{A}}_\alpha$  as follows:

$$\mathcal{I}_\mu^\alpha := (\pi_* {}^\pi\mathcal{I}_\mu)^{\text{sgn}} \quad (15)$$

where the superscript  $(\bullet)^{\text{sgn}}$  denotes the subsheaf of sections  $x$  such that  $\sigma x = \text{sgn}(\sigma)x$  for all  $\sigma \in \Sigma_\pi$ . Cf. II.8.16.

Alternatively, we can define this local system as follows. By the descent, there exists a unique local system  $\mathcal{I}'_\mu$  over  $\mathcal{A}_\alpha$  such that  $\pi^*\mathcal{I}'_\mu$  is equal to  ${}^\pi\mathcal{I}_\mu$  with the equivariant structure described above. In fact,

$$\mathcal{I}'_\mu = (\pi_* {}^\pi\mathcal{I}_\mu)^{\Sigma_\pi}$$

where the superscript  $(\bullet)^{\Sigma_\pi}$  denotes invariants. We have

$$\mathcal{I}_\mu^\alpha = \mathcal{I}'_\mu \otimes \mathcal{S} \setminus \quad (16)$$

where  $\mathcal{S} \setminus$  denotes the one-dimensional local system over  $\mathcal{A}_\alpha$  associated with the sign representation  $\pi_1(\mathcal{A}_\alpha) \rightarrow \Sigma_\pi \xrightarrow{\text{sgn}} \{\pm\infty\}$ .

This definition does not depend (up to a canonical isomorphism) upon the choice of an unfolding.

3.2. For each triple  $(\vec{\alpha}, \vec{d}, \mu)$  as in the previous section, let us denote by  $\mathcal{I}_\mu^{\vec{\alpha}}(\vec{\Gamma})$  the restriction of  $\mathcal{I}_\mu^\alpha$  to the subspace  $\overset{\circ}{\mathcal{A}}_\mu^{\vec{\alpha}}(\vec{d}) \subset \overset{\circ}{\mathcal{A}}^\alpha(\mu)$  where  $\alpha \in \mathbb{N}[\mathbb{I}]$  is the sum of components of  $\vec{\alpha}$ .

Let us define *factorization isomorphisms*

$$\phi_i = \phi_{\mu; i}^{\vec{\alpha}}(\vec{d}) : \mathcal{I}_\mu^{\vec{\alpha}}(\vec{\Gamma}) \xrightarrow{\sim} \mathcal{I}_{\mu \leq}^{\vec{\alpha}_{\leq}}(\vec{\Gamma}_{\leq}) \boxtimes \mathcal{I}_{\mu \geq}^{\vec{\alpha}_{\geq}}(\vec{\Gamma}_{\geq}) \quad (17)$$

(we are using identifications (11)). By definition, we have canonical identifications of the stalks of all three local systems over a point with real coordinates, with  $B$ . We define (17) as a unique isomorphism acting as identity when restricted to such a stalk. We will omit irrelevant indices from the notation for  $\phi$  if there is no risk of confusion.

**3.3. Associativity.** These isomorphisms have the following *associativity property*.

For all  $i < j$ , diagrams

$$\begin{array}{ccc}
 & \mathcal{I}(\vec{\Gamma}) & \\
 \phi_j \swarrow & & \searrow \phi_i \\
 \mathcal{I}(\vec{\Gamma}_{\leq i}) \boxtimes \mathcal{I}(\vec{\Gamma}_{\geq j}) & & \mathcal{I}(\vec{\Gamma}_{\leq j}) \boxtimes \mathcal{I}(\vec{\Gamma}_{\geq i}) \\
 \phi_i \boxtimes \text{id} \searrow & & \swarrow \text{id} \boxtimes \phi_j \\
 & \mathcal{I}(\vec{\Gamma}_{\leq i}) \boxtimes \mathcal{I}(\vec{\Gamma}_{\geq i, \leq j}) \boxtimes \mathcal{I}(\vec{\Gamma}_{\geq j}) &
 \end{array}$$

are commutative.

In fact, it is enough to check the commutativity restricted to some fiber  $(\bullet)_C$ , where it is obvious.

**3.4.** The collection of local systems  $\mathcal{I} = \{\mathcal{I}_\mu^\alpha\}$  together with factorization isomorphisms (17) will be called *the braiding local system* (over  $\mathring{\mathcal{A}}$ ).

The couple  $(\mathcal{A}, \mathcal{I})$  will be called *the semi-infinite configuration space associated with the Cartan datum  $(I, \cdot)$  and parameter  $\zeta$* .

**3.5.** Let  $j : \mathring{\mathcal{A}}_\mu^{\vec{\alpha}}(\vec{d}) \hookrightarrow \bullet \mathcal{A}_\mu^{\vec{\alpha}}(\vec{d})$  denote an embedding; let us define a preverse sheaf

$$\mathring{\mathcal{I}}_\mu^{\vec{\alpha}}(\vec{d}) := j_* \mathcal{I}_\mu^{\vec{\alpha}}(\vec{\Gamma})[\dim \mathcal{A}_\mu^{\vec{\alpha}}] \in \mathcal{M}(\bullet \mathcal{A}_\mu^{\vec{\alpha}}(\vec{\Gamma}); \mathcal{S})$$

By functoriality, factorization isomorphisms (17) induce analogous isomorphisms (denoted by the same letter)

$$\phi_i = \phi_{\mu; i}^{\vec{\alpha}}(\vec{d}) : \mathring{\mathcal{I}}_\mu^{\vec{\alpha}}(\vec{d}) \xrightarrow{\sim} \mathring{\mathcal{I}}_{\mu_{\leq i}}^{\vec{\alpha}_{\leq i}}(\vec{d}_{\leq i}) \boxtimes \mathring{\mathcal{I}}_\mu^{\vec{\alpha}_{\geq i}}(\vec{d}_{\geq i}) \quad (18)$$

satisfying the associativity property completely analogous to 3.3.

#### 4. FACTORIZABLE SHEAVES

**4.1.** The aim of this section is to define certain  $B$ -linear category  $\widetilde{\mathcal{FS}}$ . Its objects will be called *factorizable sheaves* (over  $(\mathcal{A}, \mathcal{I})$ ). By definition,  $\widetilde{\mathcal{FS}}$  is a direct product of  $B$ -categories  $\widetilde{\mathcal{FS}}_c$ , where  $c$  runs through  $\pi_0(\mathcal{A})$  (see 2.6). Objects of  $\widetilde{\mathcal{FS}}_c$  will be called *factorizable sheaves supported at  $\mathcal{A}_j$* .

In what follows we pick  $c$ , and denote by  $X_c \subset X$  the corresponding coset modulo  $Y$ .



**4.2. Definition.** A factorizable sheaf  $\mathcal{X}$  over  $(\mathcal{A}, \mathcal{I})$  supported at  $\mathcal{A}_\lambda$  is the following collection of data:

(a) a weight  $\lambda \in X_c$ ; it will be denoted by  $\lambda(\mathcal{X})$ ;

(b) for each  $\alpha \in \mathbb{N}[\mathbb{I}]$ , a sheaf  $\mathcal{X}^\alpha \in \mathcal{M}(\mathcal{A}_\lambda^\alpha; \mathcal{S})$ ;

we will denote by  $\mathcal{X}^{\vec{\alpha}}(\vec{\Gamma})$  perverse sheaves over  $\mathcal{A}_\lambda^{\vec{\alpha}}(\vec{\Gamma})$  obtained by taking the restrictions with respect to the embeddings  $\mathcal{A}_\lambda^{\vec{\alpha}}(\vec{\Gamma}) \hookrightarrow \mathcal{A}_\lambda^\alpha$ ;

(c) for each  $\alpha, \beta \in \mathbb{N}[\mathbb{I}]$ ,  $\alpha > \beta$ , a factorization isomorphism

$$\psi^{\alpha, \beta}(d) : \mathcal{X}^{(\alpha, \beta)}(\Gamma) \xrightarrow{\sim} \dot{\mathcal{I}}_{\lambda - \beta'}^{(\alpha, \iota)}(\Gamma) \boxtimes \mathcal{X}^{(\iota, \beta)}(\Gamma) \quad (19)$$

such that

(associativity) for each  $\alpha, \beta, \gamma \in \mathbb{N}[\mathbb{I}]$ ,  $\alpha > \beta > \gamma$ , the square below must commute:

$$\begin{array}{ccc} & \mathcal{X}^{(\alpha, \beta, \gamma)}(\Gamma_\infty, \Gamma_\epsilon) & \\ \swarrow \psi & & \searrow \psi \\ \dot{\mathcal{I}}_{\lambda - \gamma'}^{(\alpha, \beta, 0)}(d_1, d_2) \boxtimes \mathcal{X}^{(\iota, \gamma)}(\Gamma_\epsilon) & & \dot{\mathcal{I}}_{\lambda - \beta' - \gamma'}^{(\alpha, 0)}(d_1) \boxtimes \mathcal{X}^{(\iota, \beta, \gamma)}(\Gamma_\infty, \Gamma_\epsilon) \\ \searrow \phi \boxtimes \text{id} & & \swarrow \text{id} \boxtimes \psi \\ \dot{\mathcal{I}}_{\lambda - \beta' - \gamma'}^{(\alpha, 0)}(d_1) \boxtimes \dot{\mathcal{I}}_{\lambda - \gamma'}^{(0, \beta, 0)}(d_1, d_2) \boxtimes \mathcal{X}^{(\iota, \gamma)}(\Gamma_\epsilon) & & \end{array}$$

**4.2.1.** Remark that with these definitions, the braiding local system  $\mathcal{I}$  resembles a “coalgebra”, and a factorizable sheaf — a “comodule” over it.

**4.3. Remark.** Note an immediate corollary of the factorization axiom. We have isomorphisms

$$\mathcal{X}^{(\alpha, \iota)}(\Gamma) \cong \mathcal{X}' \otimes \dot{\mathcal{I}}_\lambda^{(\alpha, \iota)}(\Gamma) \quad (20)$$

(where  $\mathcal{X}'$  is simply a vector space).

Our next aim is to define morphisms between factorizable sheaves.

**4.4.** Let  $\mathcal{X}$  be as above. For each  $\mu \geq \lambda$ ,  $\mu = \lambda + \beta'$ , and  $\alpha \in \mathbb{N}[\mathbb{I}]$ , let us define a sheaf  $\mathcal{X}_\mu^\alpha \in \mathcal{M}(\mathcal{A}_\mu^\alpha; \mathcal{S})$  as  $\sigma_* \mathcal{X}^{\alpha - \beta}$ . For example,  $\mathcal{X}_\lambda^\alpha = \mathcal{X}^\alpha$ . By taking restriction, the sheaves  $\mathcal{X}_\mu^{\vec{\alpha}}(\vec{\Gamma}) \in \mathcal{M}(\mathcal{A}_\mu^{\vec{\alpha}}(\vec{\Gamma}); \mathcal{S})$  are defined.

Suppose  $\mathcal{X}, \mathcal{Y}$  are two factorizable sheaves supported at  $\mathcal{A}_J$ ,  $\lambda = \lambda(\mathcal{X})$ ,  $\nu = \lambda(\mathcal{Y})$ . Let  $\mu \in X$ ,  $\mu \geq \lambda$ ,  $\mu \geq \nu$ ,  $\alpha, \beta \in \mathbb{N}[\mathbb{I}]$ . By definition we have canonical isomorphisms

$$\theta = \theta_{\mu}^{\beta, \alpha} : \text{Hom}_{\mathcal{A}_{\mu}^{\alpha}}(\mathcal{X}_{\mu}^{\alpha}, \mathcal{Y}_{\mu}^{\alpha}) \xrightarrow{\sim} \text{Hom}_{\mathcal{A}_{\mu+\beta'}^{\alpha+\beta}}(\mathcal{X}_{\mu+\beta'}^{\alpha+\beta}, \mathcal{Y}_{\mu+\beta'}^{\alpha+\beta}) \quad (21)$$

The maps (19) induce analogous isomorphisms

$$\psi_{\mu}^{\alpha, \beta}(d) : \mathcal{X}_{\mu}^{(\alpha, \beta)}(\Gamma) \xrightarrow{\sim} \dot{\mathcal{I}}_{\mu-\beta'}^{(\alpha, \iota)}(\Gamma) \boxtimes \mathcal{X}_{\mu}^{(\iota, \beta)}(\Gamma) \quad (22)$$

which satisfy the same associativity property as in 4.2.

For  $\alpha \geq \beta$  let us define maps

$$\tau_{\mu}^{\alpha\beta} : \text{Hom}_{\mathcal{A}_{\mu}^{\alpha}}(\mathcal{X}_{\mu}^{\alpha}, \mathcal{Y}_{\mu}^{\alpha}) \longrightarrow \text{Hom}_{\mathcal{A}_{\mu}^{\beta}}(\mathcal{X}_{\mu}^{\beta}, \mathcal{Y}_{\mu}^{\beta}) \quad (23)$$

as compositions

$$\begin{aligned} & \text{Hom}_{\mathcal{A}_{\mu}^{\alpha}}(\mathcal{X}_{\mu}^{\alpha}, \mathcal{Y}_{\mu}^{\alpha}) \xrightarrow{\nabla|f} \text{Hom}_{\mathcal{A}_{\mu}^{(\alpha-\beta, \beta)}(\Gamma)}(\mathcal{X}_{\mu}^{(\alpha-\beta, \beta)}(\Gamma), \mathcal{Y}_{\mu}^{(\alpha-\beta, \beta)}(\Gamma)) \xrightarrow{\psi} \\ & \xrightarrow{\psi} \text{Hom}_{\mathcal{A}_{\mu-\beta'}^{(\alpha-\beta, \iota)}(\Gamma)}(\dot{\mathcal{I}}_{\mu-\beta'}^{(\alpha-\beta, 0)}(d), \dot{\mathcal{I}}_{\mu-\beta'}^{(\alpha-\beta, 0)}(d)) \otimes \text{Hom}_{\mathcal{A}_{\mu}^{(\iota, \beta)}(\Gamma)}(\mathcal{X}_{\mu}^{(\iota, \beta)}(\Gamma), \mathcal{Y}_{\mu}^{(\iota, \beta)}(\Gamma)) = \\ & = B \otimes_B \text{Hom}_{\mathcal{A}_{\mu}^{(\iota, \beta)}(\Gamma)}(\mathcal{X}_{\mu}^{(\iota, \beta)}(\Gamma), \mathcal{Y}_{\mu}^{(\iota, \beta)}(\Gamma)) = \text{Hom}_{\mathcal{A}_{\mu}^{(\iota, \beta)}(\Gamma)}(\mathcal{X}_{\mu}^{(\iota, \beta)}(\Gamma), \mathcal{Y}_{\mu}^{(\iota, \beta)}(\Gamma)) \xrightarrow{\sim} \\ & \xrightarrow{\sim} \text{Hom}_{\mathcal{A}_{\mu}^{\beta}}(\mathcal{X}_{\mu}^{\beta}, \mathcal{Y}_{\mu}^{\beta}) \end{aligned} \quad (24)$$

where we have chosen some  $d > 0$ , the first map is the restriction, the second one is induced by the factorization isomorphism, the last one is inverse to the restriction. This definition does not depend on the choice of  $d$ .

The associativity axiom implies that these maps satisfy an obvious transitivity property. They are also compatible in the obvious way with the isomorphisms  $\theta$ .

We define the space  $\text{Hom}_{\widetilde{\mathcal{FS}}_c}(\mathcal{X}, \mathcal{Y})$  as the following inductive-projective limit

$$\text{Hom}_{\widetilde{\mathcal{FS}}_c}(\mathcal{X}, \mathcal{Y}) := \lim_{\rightarrow \mu} \lim_{\leftarrow \beta} \text{Hom}(\mathcal{X}_{\mu}^{\beta}, \mathcal{Y}_{\mu}^{\beta}) \quad (25)$$

where the inverse limit is over  $\beta \in \mathbb{N}[\mathbb{I}]$ , the transition maps being  $\tau_{\mu}^{\alpha\beta}$ ,  $\mu$  being fixed, and the direct limit over  $\mu \in X$  such that  $\mu \geq \lambda$ ,  $\mu \geq \nu$ , the transition maps being induced by (21).

With these spaces of homomorphisms, factorizable sheaves supported at  $\mathcal{A}_J$  form a  $B$ -linear category to be denoted by  $\widetilde{\mathcal{FS}}_c$  (the composition of morphisms is obvious).

As we have already mentioned, the category  $\widetilde{\mathcal{FS}}$  is by definition a product  $\prod_{c \in \pi_0(\mathcal{A})} \widetilde{\mathcal{FS}}_c$ . Thus, an object  $\mathcal{X}$  of  $\widetilde{\mathcal{FS}}$  is a direct sum  $\bigoplus_{c \in \pi_0(\mathcal{A})} \mathcal{X}_J$ , where  $\mathcal{X}_J \in \widetilde{\mathcal{FS}}_J$ . If  $\mathcal{X} \in \widetilde{\mathcal{FS}}_J$ ,  $\mathcal{Y} \in \widetilde{\mathcal{FS}}_{J'}$ , then

$$\text{Hom}_{\widetilde{\mathcal{FS}}}(\mathcal{X}, \mathcal{Y}) = \text{Hom}_{\widetilde{\mathcal{FS}}_J}(\mathcal{X}, \mathcal{Y})$$

if  $c = c'$ , and 0 otherwise.

4.5. Let  $\mathcal{V}] \sqcup_{\{}$  denote the category of finite dimensional  $B$ -vector spaces. Recall that in II.7.14 the functors of "vanishing cycles at the origin"

$$\Phi_{\alpha} : \mathcal{M}(\mathcal{A}_{\mu}^{\alpha}; \mathcal{S}) \longrightarrow \mathcal{V}] \sqcup_{\{}$$

have been defined.

Given  $\mathcal{X} \in \widetilde{\mathcal{FS}}_{\downarrow}$ , let us define for each  $\lambda \in X_c$  a vector space

$$\Phi_{\lambda}(\mathcal{X}) := \Phi_{\alpha}(\mathcal{X}_{\lambda+\alpha'}^{\alpha}) \quad (26)$$

where  $\alpha \in \mathbb{N}[\mathbb{I}]$  is such that  $\lambda + \alpha' \geq \lambda(\mathcal{X})$ . If  $\lambda \in X - X_c$ , we set  $\Phi_{\lambda}(\mathcal{X}) = \iota$ .

Due to the definition of the sheaves  $\mathcal{X}_{\mu}^{\alpha}$ , 4.4, this vector space does not depend on a choice of  $\alpha$ , up to a unique isomorphism.

This way we get an exact functor

$$\Phi : \widetilde{\mathcal{FS}}_c \longrightarrow \mathcal{V}] \sqcup_{\{}^{\mathcal{X}} \quad (27)$$

to the category of  $X$ -graded vector spaces with finite dimensional components which induces an exact functor

$$\Phi : \widetilde{\mathcal{FS}} \longrightarrow \mathcal{V}] \sqcup_{\{}^{\mathcal{X}} \quad (28)$$

4.6. **Lemma.** *If  $\Phi(\mathcal{X}) = \iota$  then  $\mathcal{X} = \iota$ .*

**Proof.** We may suppose that  $\mathcal{X} \in \widetilde{\mathcal{FS}}_{\downarrow}$  for some  $c$ . Let  $\lambda = \lambda(\mathcal{X})$ . Let us prove that for every  $\alpha = \sum a_i i \in \mathbb{N}[\mathbb{I}]$ ,  $\mathcal{X}^{\alpha} = \iota$ . Let us do it by induction on  $|\alpha| := \sum a_i$ . We have  $\mathcal{X}' = \Phi_{\lambda}(\mathcal{X}) = \iota$  by assumption.

Given an arbitrary  $\alpha$ , it is easy to see from the factorizability and induction hypothesis that  $\mathcal{X}^{\alpha}$  is supported at the origin of  $\mathcal{A}_{\alpha}$ . Since  $\Phi_{\lambda-\alpha}(\mathcal{X}) = \iota$ , we conclude that  $\mathcal{X}^{\alpha} = \iota$ .  $\square$

## 5. FINITE SHEAVES

5.1. **Definition.** *A factorizable sheaf  $\mathcal{X}$  is called finite if  $\Phi(X)$  is finite dimensional.*

This is equivalent to saying that there exists only finite number of  $\alpha \in \mathbb{N}[\mathbb{I}]$  such that  $\Phi_{\alpha}(\mathcal{X}^{\alpha}) \neq \iota$  (or  $SS(\mathcal{X}^{\alpha})$  contains the conormal bundle to the origin  $0 \in \mathcal{A}_{\lambda}^{\alpha}$ , where  $\lambda := \lambda(\mathcal{X})$ ).

5.2. **Definition.** *The category of finite factorizable sheaves (FFS for short) is a full subcategory  $\mathcal{FS} \subset \widetilde{\mathcal{FS}}$  whose objects are finite factorizable sheaves.*

We set  $\mathcal{FS}_{\downarrow} := \mathcal{FS} \cap \widetilde{\mathcal{FS}}_{\downarrow}$  for  $c \in \pi_0(\mathcal{A})$ .

This category is our main character. It is clear that  $\mathcal{FS}$  is a strictly full subcategory of  $\widetilde{\mathcal{FS}}$  closed with respect to taking subobjects and quotients.

The next stabilization lemma is important.

**5.3. Lemma.** *Let  $\mathcal{X}, \mathcal{Y}$  be two FFS's supported at the same connected component of  $\mathcal{A}$ . For a fixed  $\mu \geq \lambda(\mathcal{X}), \lambda(\mathcal{Y})$  there exists  $\alpha \in \mathbb{N}[\mathbb{I}]$  such that for any  $\beta \geq \alpha$  the transition map*

$$\tau_\mu^{\beta\alpha} : \text{Hom}_{\mathcal{A}_\mu^\beta}(\mathcal{X}_\mu^\beta, \mathcal{Y}_\mu^\beta) \longrightarrow \text{Hom}_{\mathcal{A}_\mu^\alpha}(\mathcal{X}_\mu^\alpha, \mathcal{Y}_\mu^\alpha)$$

*is an isomorphism.*

**Proof.** Let us introduce a finite set

$$N_\mu(\mathcal{Y}) := \{\alpha \in \mathbb{N}[\mathbb{I}] \mid \Phi_\alpha(\mathcal{Y}_\mu^\alpha) \neq \iota\}.$$

Let us pick  $\beta \in \mathbb{N}[\mathbb{I}]$ . Consider a non-zero map  $f : \mathcal{X}_\mu^\beta \longrightarrow \mathcal{Y}_\mu^\beta$ . For each  $\alpha \leq \beta$  we have a map  $f^\alpha := \tau_\mu^{\beta\alpha}(f) : \mathcal{X}_\mu^\alpha \longrightarrow \mathcal{Y}_\mu^\alpha$ . Let us consider subsheaves  $\mathcal{Z}^\alpha := \text{Im}(\{f^\alpha\}) \subset \mathcal{Y}_\mu^\alpha$ . These subsheaves satisfy an obvious factorization property.

Let us consider the toric stratification of  $\mathcal{A}_\mu^\beta$ . For each  $\alpha \leq \beta$  set  $\mathcal{A}^\alpha := \sigma(\mathcal{A}_{\mu+\alpha'-\beta'}^\alpha) \subset \mathcal{A}_\mu^\beta$ ;  $\dot{\mathcal{A}}^\alpha := \sigma(\dot{\mathcal{A}}_{\mu+\alpha'-\beta'}^\alpha)$ . Thus, the subspaces  $\dot{\mathcal{A}}^\alpha$  are strata of the toric stratification. We have  $\alpha_1 \leq \alpha_2$  iff  $\mathcal{A}^{\alpha_\infty} \subset \mathcal{A}^{\alpha_\epsilon}$ .

Let  $\gamma$  denote a maximal element in the set  $\{\alpha \mid \mathcal{Z}^\beta|_{\dot{\mathcal{A}}^\alpha} \neq \iota\}$ . Then it is easy to see that  $\mathcal{Z}^{\beta-\gamma}$  is a non-zero scyscraper on  $\mathcal{A}^{\beta-\gamma}$  supported at the origin. Therefore,  $\Phi_{\beta-\gamma}(\mathcal{Z}^{\beta-\gamma}) \neq \iota$ , whence  $\Phi_{\beta-\gamma}(\mathcal{Y}^{\beta-\gamma}) \neq \iota$ , i.e.  $\beta - \gamma \in N_\mu(\mathcal{Y})$ .

Suppose that for some  $\alpha \leq \beta$ ,  $\tau_\mu^{\beta\alpha}(f) = 0$ . Then

$$\mathcal{Z}^\beta|_{\mathcal{A}^{(\beta-\alpha, \alpha)}(\Gamma)} = \iota$$

for every  $d > 0$ . It follows that if  $\mathcal{Z}^\beta|_{\dot{\mathcal{A}}^\delta} \neq \iota$  then  $\delta \not\geq \beta - \alpha$ .

Let us apply this remark to  $\delta$  equal to  $\gamma$  above. Suppose that  $\gamma = \sum c_i i$ ,  $\beta = \sum b_i i$ ,  $\alpha = \sum a_i i$ . There exists  $i$  such that  $c_i < b_i - a_i$ . Recall that  $\beta - \gamma \in N_\mu(\mathcal{Y})$ . Consequently, we have

**5.3.1. Corollary.** *Suppose that  $\alpha \geq \delta$  for all  $\delta \in N_\mu(\mathcal{Y})$ . Then all the maps*

$$\tau_\mu^{\beta\alpha} : \text{Hom}(\mathcal{X}_\mu^\beta, \mathcal{Y}_\mu^\beta) \longrightarrow \text{Hom}(\mathcal{X}_\mu^\alpha, \mathcal{Y}_\mu^\alpha),$$

*$\beta \geq \alpha$ , are injective.  $\square$*

Since all the spaces  $\text{Hom}(\mathcal{X}_\mu^\alpha, \mathcal{Y}_\mu^\alpha)$  are finite dimensional due to the constructibility of our sheaves, there exists an  $\alpha$  such that all  $\tau_\mu^{\beta\alpha}$  are isomorphisms. Lemma is proven.  $\square$

**5.4.** For  $\lambda \in X_c$  let us denote by  $\mathcal{FS}_{\mathbb{J}; \leq \lambda} \subset \mathcal{FS}_{\mathbb{J}}$  the full subcategory whose objects are FFS's  $\mathcal{X}$  such that  $\lambda(\mathcal{X}_{\mathbb{J}}) \leq \lambda$ . Obviously  $\mathcal{FS}_{\mathbb{J}}$  is a filtered union of these subcategories.

We have obvious functors

$$p_\lambda^\beta : \mathcal{FS}_{\mathbb{J}; \leq \lambda} \longrightarrow \mathcal{M}(\mathcal{A}_\lambda^\beta; \mathcal{S}), \quad \mathcal{X} \mapsto \mathcal{X}_\lambda^\beta \quad (29)$$

The previous lemma claims that for every  $\mathcal{X}, \mathcal{Y} \in \mathcal{FS}_{\leq \lambda}$  there exists  $\alpha \in \mathbb{N}[\mathbb{I}]$  such that for every  $\beta \geq \alpha$  the map

$$p_\lambda^\beta : \text{Hom}_{\mathcal{FS}}(\mathcal{X}, \mathcal{Y}) \longrightarrow \text{Hom}_{\mathcal{A}_\lambda^\beta}(\mathcal{X}_\lambda^\beta, \mathcal{Y}_\lambda^\beta)$$

is an isomorphism. (Obviously, a similar claim holds true for any finite number of FFS's.)

**5.5. Lemma.**  *$\mathcal{FS}$  is an abelian artinian category.*

**Proof.**  $\mathcal{FS}$  is abelian by Stabilization lemma. Each object has finite length by Lemma 4.6.  $\square$

## 6. STANDARD SHEAVES

6.1. For  $\Lambda \in X$ , let us define factorizable sheaves  $\mathcal{M}(\Lambda)$ ,  $\mathcal{DM}(\Lambda)_{\zeta^{-\infty}}$  and  $\mathcal{L}(\Lambda)$  as follows. (The notation  $\mathcal{DM}(\Lambda)_{\zeta^{-\infty}}$  will be explained in 13.3 below).

Set

$$\lambda(\mathcal{M}(\Lambda)) = \lambda(\mathcal{DM}(\Lambda)_{\zeta^{-\infty}}) = \lambda(\mathcal{L}(\Lambda)) = \Lambda.$$

For  $\alpha \in \mathbb{N}[\mathbb{I}]$  let  $j$  denote the embedding  $\dot{\mathcal{A}}_\alpha \hookrightarrow \mathcal{A}_\alpha$ . We define

$$\mathcal{M}(\Lambda)^\alpha = |_{\dot{\mathcal{I}}_\Lambda}^\alpha; \quad \mathcal{DM}(\Lambda)_{\zeta^{-\infty}}^\alpha = |_{*\dot{\mathcal{I}}_\Lambda}^\alpha; \quad \mathcal{L}(\Lambda)^\alpha = |_{!\dot{\mathcal{I}}_\Lambda}^\alpha.$$

The factorization isomorphisms are defined by functoriality from these isomorphisms for  $\dot{\mathcal{I}}$ .

Thus, the collections  $\{\mathcal{M}(\Lambda)^\alpha\}_\alpha$ , etc. form factorizable sheaves to be denoted by  $\mathcal{M}(\Lambda)$ ,  $\mathcal{DM}(\Lambda)_{\zeta^{-\infty}}$  and  $\mathcal{L}(\Lambda)$  respectively. Obviously, we have a canonical morphism

$$m : \mathcal{M}(\Lambda) \longrightarrow \mathcal{DM}(\Lambda)_{\zeta^{-\infty}} \tag{30}$$

and  $\mathcal{L}(\Lambda)$  is equal to its image.

**6.2. Theorem.** (i) *The factorizable sheaves  $\mathcal{L}(\Lambda)$  are finite.*

(ii) *They are irreducible objects of  $\mathcal{FS}$ , non-isomorphic for different  $\Lambda$ , and they exhaust all irreducibles in  $\mathcal{FS}$ , up to isomorphism.*

**Proof.** (i) follows from II.8.18.

(ii) Since the sheaves  $\mathcal{L}(\Lambda)^\alpha$  are irreducible as objects of  $\mathcal{M}(\mathcal{A}_\alpha; \mathcal{S})$ , the irreducibility of  $\mathcal{L}(\Lambda)$  follows easily. It is clear that they are non-isomorphic (consider the highest component).

Suppose  $\mathcal{X}$  is an irreducible FFS,  $\lambda = \lambda(\mathcal{X})$ . Let  $\alpha \in \mathbb{N}[\mathbb{X}]$  be a minimal among  $\beta$  such that  $\Phi_{\lambda-\beta}(\mathcal{X}) \neq \iota$ ; set  $\Lambda = \lambda - \alpha$ . By factorizability and the universal property of  $!$ -extension, there exists a morphism if FS's  $f : \mathcal{M}(\Lambda) \longrightarrow \mathcal{X}$  such that  $\Phi_\Lambda(f) \neq 0$  (hence is a monomorphism). It follows from irreducibility of  $\mathcal{L}(\Lambda)$  that the composition

$\text{Ker}(f) \longrightarrow \mathcal{M}(\Lambda) \longrightarrow \mathcal{L}(\Lambda)$  is equal to zero, hence  $f$  factors through a non-zero morphism  $\mathcal{L}(\Lambda) \longrightarrow \mathcal{X}$  which must be an isomorphism.  $\square$

6.3. Let us look more attentively at the sheaf  $\mathcal{L}(\iota)$ .

Let  $\tilde{\mathcal{A}}^\alpha \subset \mathcal{A}^\alpha$  denote the open stratum of the *diagonal* stratification, i.e. the complement to the diagonals. Thus,  $\overset{\circ}{\mathcal{A}}^\alpha \subset \tilde{\mathcal{A}}^\alpha$ . Let  $\tilde{\mathcal{I}}^\alpha$  denote the local system over  $\tilde{\mathcal{A}}^\alpha$  defined in the same way as local systems  $\mathcal{I}_\mu^\alpha$ , but using only "diagonal" monodromies, cf. II.6.3.

One sees immediately that  $\mathcal{L}(\iota)^\alpha$  is equal to the middle extension of  $\tilde{\mathcal{I}}^\alpha$ .

## CHAPTER 2. Tensor structure.

### 7. MARKED DISK OPERAD

7.1. Let  $K$  be a finite set. If  $T$  is any set, we will denote by  $T^K$  the set of all mappings  $K \longrightarrow T$ ; elements of  $T^K$  will be denoted typically by  $\vec{x} = (x_k)_{k \in K}$ .

We will use the following partial orders on  $X^K, \mathbb{N}[\mathbb{I}]^K$ . For  $\vec{\lambda} = (\lambda_k), \vec{\mu} = (\mu_k) \in X^K$ , we write  $\vec{\lambda} \geq \vec{\mu}$  iff  $\lambda_k \geq \mu_k$  for all  $k$ . An order on  $\mathbb{N}[\mathbb{I}]^K$  is defined in the same manner.

For  $\vec{\alpha} = (\alpha_k) \in \mathbb{N}[\mathbb{I}]^K$  we will use the notation  $\alpha$  for the sum of its components  $\sum_{k \in K} \alpha_k$ ; the same agreement will apply to  $X^K$ .

$\mathbb{A}^K$  will denote the complex affine space with fixed coordinates  $u_k, k \in K$ ;  $\overset{\circ}{\mathbb{A}}^K \subset \mathbb{A}^K$  will denote the open stratum of the diagonal stratification.

7.2. **Trees.** We will call a *tree* a couple

$$\tau = (\sigma, \vec{d}) \tag{31}$$

where  $\sigma$ , to be called *the shape* of  $\tau$ ,

$$\sigma = (K_p \xrightarrow{\rho_{p-1}} K_{p-1} \xrightarrow{\rho_{p-2}} \dots \xrightarrow{\rho_1} K_1 \xrightarrow{\rho_0} K_0)$$

is a sequence of epimorphisms of finite sets, such that  $\text{card}(K_0) = 1$ ,  $\vec{d} = (d_0, d_1, \dots, d_p)$ , to be called *the thickness* of  $\tau$  — a tuple of real numbers such that  $d_0 = 1 > d_1 > \dots > d_p \geq 0$ .

We will use a notation  $\rho_{ab}$  for composition  $K_a \longrightarrow K_{a-1} \longrightarrow \dots \longrightarrow K_b, a > b$ .

A number  $p \geq 0$  will be called *the height* of  $\tau$  and denoted  $\text{ht}(\tau)$ . Elements  $k \in K_i$  will be called *branches of height  $i$* ;  $d_i$  will be called *the thickness of  $k$* . A unique branch of height 0 will be called *bole* and denoted by  $\ast(\tau)$ .

The set  $K_p$  will be called *the base* of  $\tau$  and denoted  $K_\tau$ ; we will also say that  $\tau$  is  *$K_p$ -based*; we will denote  $d_p$  by  $d_\tau$ . We will use notation  $K(\tau)$  for the set  $\coprod_{i=0}^p K_i$  and  $K'(\tau)$  for  $\coprod_{i=0}^{p-1} K_i$ .

A tree of height one will be called *elementary*. A tree  $\tau$  whose branches of height  $\text{ht}(\tau)$  have thickness 0, will be called *grown up*; otherwise it will be called *young*. We will assign to every tree  $\tau$  a grown up tree  $\tilde{\tau}$  by changing the thickness of the thinnest branches to zero.

Thus, an elementary tree is essentially a finite set and a real  $0 \leq d < 1$ ; a grown up elementary tree is essentially a finite set.

7.2.1. *Cutting.* Suppose we have a tree  $\tau$  as above, and an integer  $i$ ,  $0 < i < p$ . We define the operation of *cutting*  $\tau$  at level  $i$ . It produces from  $\tau$  new trees  $\tau_{\leq i}$  and  $\tau_{\geq k}$ ,  $k \in K_i$ . Namely,

$$\tau_{\leq i} = (\sigma_{\leq i}, \vec{d}_{\leq i}),$$

where  $\sigma_{\leq i} = (K_i \longrightarrow K_{i-1} \longrightarrow \dots \longrightarrow K_0)$  and  $\vec{d}_{\leq i} = (d_0, d_1, \dots, d_i)$ .

Second, for  $k \in K_i$

$$\tau_{\geq k} = (\sigma_{\geq k}, \vec{d}_{> i})$$

where

$$\begin{aligned} \sigma_{\geq k} &= (\rho_{pi}^{-1}(k) \longrightarrow \rho_{p-1,i}^{-1}(k) \longrightarrow \dots \longrightarrow \rho_i^{-1}(k) \longrightarrow \{k\}), \\ d_{> i} &= (1, d_{i+1}d_i^{-1}, \dots, d_p d_i^{-1}). \end{aligned}$$

7.2.2. For  $0 < i \leq p$  we will denote by  $\partial_i \tau$  a tree  $(\partial_i \sigma, \partial_i \vec{d})$  where

$$\partial_i \sigma = (K_p \longrightarrow \dots \longrightarrow \hat{K}_i \longrightarrow \dots \longrightarrow K_0),$$

and  $\partial_i \vec{d}$  is obtained from  $\vec{d}$  by omitting  $d_i$ .

7.3. **Operad of disks.** For  $r \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ ,  $F \in \mathbb{C}$ , we define an open disk  $D(z; r) := \{u \in \mathbb{A}^{\mathbb{K}} \mid |\cong - F| < r\}$ , and a closed disk  $\bar{D}(z; r) := \{u \in \mathbb{A}^{\mathbb{K}} \mid |\cong - F| \leq r\}$ .

For a tree (31) we define a space

$$\mathcal{O}(\tau) = \mathcal{O}(\sigma; \vec{d})$$

parametrizing all collections  $\vec{D} = (\bar{D}_k)_{k \in K_\tau}$  of closed disks, such that  $D_{*(\tau)} = D(0; 1)$ , for  $k \in K_i$  the disk  $\bar{D}_k$  has radius  $d_i$ , for fixed  $i \in [p]$  the disks  $\bar{D}_k$ ,  $k \in K_i$ , do not intersect, and for each  $i \in [0, p-1]$  and each  $k \in K_{i+1}$  we have  $\bar{D}_k \subset D_{\rho_i(k)}$ .

Sometimes we will call such a collection *a configuration of disks shaped by a tree  $\tau$* .

7.3.1. Given such a configuration, we will use the notation

$$\overset{\circ}{D}_k(\tau) = D_k - \bigcup_{l \in \rho_i^{-1}(k)} (\bar{D}_l) \quad (32)$$

if  $k \in K_i$  and  $i < p$ , and we set  $\overset{\circ}{D}_k(\tau) = D_k$  if  $i = p$ .

If  $\tau = (K \longrightarrow \{*\}; d)$  is an elementary tree, we will use the notation  $\mathcal{O}(\mathcal{K}; \vec{d})$  for  $\mathcal{O}(\tau)$ ; if  $d = 0$ , we will abbreviate the notation to  $\mathcal{O}(\mathcal{K})$ .

We have obvious embeddings

$$\mathcal{O}(\mathcal{K}; \vec{d}) \hookrightarrow \mathcal{O}(\mathcal{K}) \quad (33)$$

and

$$\mathcal{O}(\mathcal{K}) \hookrightarrow \overset{\circ}{\mathbb{A}}^{\mathcal{K}} \quad (34)$$



this one is a homotopy equivalence.

We have open embeddings

$$\mathcal{O}(\tau) \hookrightarrow \mathcal{O}(\tilde{\tau}) \quad (35)$$

obtained by changing the radius of smallest discs to zero.

**7.3.2. Substitution.** For each tree  $\tau$  and  $0 < i < \text{ht}(\tau)$  we have the following *substitution isomorphisms*

$$\mathcal{O}(\tau) \cong \mathcal{O}(\tau_{\leq i}) \times \prod_{\parallel \in \mathcal{K}_i} \mathcal{O}(\tau_{\geq \parallel}) \quad (36)$$

In fact, a configuration of disks shaped by a tree  $\tau$  is the same as a configuration shaped by  $\tau_{\leq i}$ , and for each  $k \in K_i$  a configuration shaped by  $\tau_{\geq k}$  inside  $D_k$  (playing the role of  $D_0$ ; here we have to make a dilation by  $d_i^{-1}$ ).

These isomorphisms satisfy obvious quadratic relations connected with pairs  $0 < i < j < \text{ht}(\tau)$ . We leave their formulation to the reader.

**7.4. Enhanced trees.** We will call an *enhanced tree* a couple  $(\tau, \vec{\alpha})$  where  $\tau$  is a tree and  $\vec{\alpha} \in \mathbb{N}[\mathbb{I}]^{\mathbb{K}'(\tau)}$ . Vector  $\vec{\alpha}$  will be called *enhancement* of  $\tau$ .

Let us define cutting for enhanced trees. Given  $\tau$  and  $i$  as in 7.2.1, let us note that  $K'(\tau_{\leq i})$  and  $K'(\tau_{\geq k})$  are subsets of  $K'(\tau)$ . We define

$$\vec{\alpha}_{\leq i} \in \mathbb{N}[\mathbb{I}]^{\mathbb{K}'(\tau_{\leq i})}, \quad \vec{\alpha}_{\geq \gamma} \in \mathbb{N}[\mathbb{I}]^{\mathbb{K}'(\tau_{\geq \gamma})}$$

as the corresponding subsequences of  $\vec{\alpha}$ .

Let us define operations  $\partial_i$  for enhanced trees. Namely, in the setup of 7.2.2, we define  $\partial_i \vec{\alpha} = (\alpha'_k) \in \mathbb{N}[\mathbb{I}]^{\mathbb{K}'(\partial_i \tau)}$  as follows. If  $i = p$  then  $K'(\partial_p \tau) \subset K'(\tau)$ , and we define  $\partial_p \vec{\alpha}$  as a corresponding subsequence. If  $i < p$ , we set  $\alpha'_k = \alpha_k$  if  $k \in K_j$ ,  $j > i$  or  $j < i - 1$ . If  $j = i - 1$ , we set

$$\alpha'_k = \alpha_k + \sum_{l \in \rho_{i-1}^{-1}(k)} \alpha_l.$$

**7.5. Enhanced disk operad.** Given an enhanced tree  $(\tau, \vec{\alpha})$ , let us define a configuration space  $\mathcal{A}^{\vec{\alpha}}(\tau)$  as follows. Its points are couples  $(\vec{D}, \mathbf{t})$ , where  $\vec{D} \in \mathcal{O}(\tau)$  and  $\mathbf{t} = (t_j)$  is an  $\alpha$ -colored configuration in  $\mathbb{A}^{\mathbb{K}}$  (see II.6.12) such that

for each  $k \in K'(\tau)$  exactly  $\alpha_k$  points lie inside  $\overset{\circ}{D}_k(\tau)$  if  $k \notin K_\tau$  (resp., inside  $D_k(\tau)$  if  $k \in K_\tau$ ) (see 7.3.1).

In particular, all points lie inside  $D_{*(\tau)} = D(0; 1)$  and outside  $\bigcup_{k \in K_\tau} \bar{D}_k$  if  $\tau$  is young. This space is an open subspace of the product  $\mathcal{O}(\tau) \times \mathcal{A}_\alpha$ .

We will also use a notation

$$\mathcal{A}^\alpha(\mathcal{L}; \sqcap) := \mathcal{A}^\alpha(\mathcal{L} \longrightarrow \{*\}; \sqcap)$$

for elementary trees and  $\mathcal{A}^\alpha(\mathcal{L})$  for  $\mathcal{A}^\alpha(\mathcal{L}; \iota)$ .

The isomorphisms (36) induce isomorphisms

$$\mathcal{A}^{\vec{\alpha}}(\tau) \cong \mathcal{A}^{\vec{\alpha} \leq}(\tau_{\leq}) \times \prod_{\parallel \in \mathcal{K}_\gamma} \mathcal{A}^{\vec{\alpha} \geq \parallel}(\tau_{\geq \parallel}) \quad (37)$$

We have embeddings

$$d_i : \mathcal{A}^{\vec{\alpha}}(\tau) \longrightarrow \mathcal{A}^{\partial_i \vec{\alpha}}(\partial_i \tau), \quad \iota < \rangle \leq \sqrt{\phantom{x}} \quad (38)$$

— dropping all disks  $D_k$ ,  $k \in K_i$ .

We have obvious open embeddings

$$\mathcal{A}^{\vec{\alpha}}(\tau) \hookrightarrow \mathcal{A}^\alpha(\mathcal{K}_\tau; \lceil_\tau) \hookrightarrow \mathcal{A}^\alpha(\mathcal{K}_\tau) \quad (39)$$

**7.6. Marked trees.** We will call a *marked tree* a triple  $(\tau, \vec{\alpha}, \vec{\mu})$  where  $(\tau, \vec{\alpha})$  is an enhanced tree, and  $\vec{\mu} \in X^{K_\tau}$ . We will call  $\text{ht}(\tau)$  the *height* of this marked tree.

Let us define operations  $\partial_i$ ,  $0 < i \leq p = \text{ht}(\tau)$  for marked trees. Namely, for  $i < p$  we set  $\partial_i \vec{\mu} = \vec{\mu}$ . For  $i = p$  we define  $\partial_p \vec{\mu}$  as  $(\mu'_k)_{k \in K_{p-1}}$ , where

$$\mu'_k = \sum_{l \in \rho_{p-1}^{-1}(k)} \mu_l - \alpha'_l.$$

Let us define cutting for marked trees. Namely, for  $1 \leq i < p$  we define  $\vec{\mu}_{\leq i}$  as  $\partial_{i+1} \dots \partial_{p-1} \partial_p \vec{\mu}$ .

Next, for  $k \in K_i$  we have  $K_{\tau_{\geq k}} \subset K_\tau$ , and we define  $\vec{\mu}_{\geq k}$  as a corresponding subsequence of  $\vec{\mu}$ .

**7.7. Marked disk operad.** Now we can introduce our main objects. For each marked tree  $(\tau, \vec{\alpha}, \vec{\mu})$  we define  $\mathcal{A}_{\vec{\mu}}^{\vec{\alpha}}(\tau)$  as a topological space  $\mathcal{A}^{\vec{\alpha}}(\tau)$  defined above, together with a marking  $\vec{\mu}$  of the tree  $\tau$  considered as an additional index assigned to this space.

We will regard  $\mathcal{A}_{\vec{\mu}}^{\vec{\alpha}}(\tau)$  as a space whose points are configurations  $(\vec{D}, \mathbf{t}) \in \mathcal{A}^{\vec{\alpha}}(\tau)$ , together with a marking of smallest disks  $D_k$ ,  $k \in K_\tau$ , by weights  $\mu_k$ .

As above, we will use abbreviations  $\mathcal{A}_{\vec{\mu}}^\alpha(\mathcal{L}; \lceil)$  for  $\mathcal{A}_{\vec{\mu}}^\alpha(\mathcal{L} \longrightarrow \{*\}; \lceil)$  (where  $\vec{\mu} \in X^L$ ) and  $\mathcal{A}_{\vec{\mu}}^\alpha(\mathcal{L})$  for  $\mathcal{A}_{\vec{\mu}}^\alpha(\mathcal{L}; \iota)$ .

We have natural open embeddings

$$d_i : \mathcal{A}_{\vec{\mu}}^{\vec{\alpha}} \longrightarrow \mathcal{A}_{\partial_i \vec{\mu}}^{\partial_i \vec{\alpha}}(\partial_i \tau), \quad \iota < \rangle \leq \sqrt{\phantom{x}} \quad (40)$$

and

$$\mathcal{A}^{\vec{\alpha}}(\tau) \hookrightarrow \mathcal{A}^\alpha(\mathcal{K}_\tau; \lceil_\tau) \hookrightarrow \mathcal{A}^\alpha(\mathcal{K}_\tau) \quad (41)$$

induced by the corresponding maps without marking.

The substitution isomorphisms (37) induce isomorphisms

$$\mathcal{A}_{\vec{\mu}}^{\vec{\alpha}}(\tau) \cong \mathcal{A}_{\vec{\mu}_{\leq}}^{\vec{\alpha}_{\leq}}(\tau_{\leq}) \times \prod_{\|\in \mathcal{K}_{\geq}} \mathcal{A}_{\vec{\mu}_{\geq}}^{\vec{\alpha}_{\geq}}(\tau_{\geq}) \quad (42)$$

7.8. We define closed embeddings

$$\sigma = \sigma_{\vec{\mu}; \vec{\beta}}^{\alpha} : \mathcal{A}_{\vec{\mu}}^{\alpha}(\mathcal{K}) \longrightarrow \mathcal{A}_{\vec{\mu} + \vec{\beta}}^{\alpha + \beta}(\mathcal{K}) \quad (43)$$

where  $\vec{\beta} = (\beta_k)_{k \in K}$ ,  $\beta_k = \sum_i b_k^i \cdot i$  and  $\beta = \sum_k \beta_k$ . By definition,  $\sigma$  leaves points  $u_k$  intact (changing their markings) and adds  $b_k^i$  copies of points of color  $i$  equal to  $u_k$ .

7.9. **Stratifications.** We set

$$\dot{\mathcal{A}}_{\vec{\mu}}^{\alpha}(K) := \mathcal{A}_{\vec{\mu}}^{\alpha}(\mathcal{K}) - \bigcup_{\vec{\gamma} > \alpha} \sigma(\mathcal{A}_{\vec{\mu} - \vec{\gamma}}^{\alpha - \vec{\gamma}}(\mathcal{K}))$$

We define a *toric stratification* of  $\mathcal{A}_{\vec{\mu}}^{\alpha}(\mathcal{K})$  as

$$\mathcal{A}_{\vec{\mu}}^{\alpha}(\mathcal{K}) = \prod_{\vec{\beta} < \alpha} \sigma(\dot{\mathcal{A}}_{\vec{\mu} - \vec{\alpha}' + \vec{\beta}'}^{\beta}(\mathcal{K}))$$

A *principal stratification* on  $\mathcal{A}_{\vec{\mu}}^{\alpha}(\mathcal{K})$  is defined as follows. The space  $\mathcal{A}_{\vec{\mu}}^{\alpha}(\mathcal{K})$  is a quotient of  $\overset{\circ}{\mathbb{A}}^K \times \mathbb{A}^{\mathbb{J}}$  where  $\pi : J \longrightarrow I$  is an unfolding of  $\alpha$  (cf. II.6.12). We define the principal stratification as the image of the diagonal stratification on  $\overset{\circ}{\mathbb{A}}^K \times \mathbb{A}^{\mathbb{J}}$  under the canonical projection  $\overset{\circ}{\mathbb{A}}^K \times \mathbb{A}^{\mathbb{J}} \longrightarrow \mathcal{A}_{\vec{\mu}}^{\alpha}(\mathcal{K})$ . We will denote by  $\dot{\mathcal{A}}_{\vec{\mu}}^{\alpha}(K)$  the open stratum of the principal stratification.

## 8. COHESIVE LOCAL SYSTEMS ${}^K\mathcal{I}$

8.1. Let us fix a non-empty finite set  $K$ . Suppose we are given  $\vec{\mu} \in X^K$  and  $\alpha \in \mathbb{N}[\mathbb{J}]$ . Let us pick an unfolding of  $\alpha$ ,  $\pi : J \longrightarrow I$ . Let

$$\pi_{\vec{\mu}}^{\alpha} : (D(0; 1)^K \times D(0; 1)^J)^{\circ} \longrightarrow \dot{\mathcal{A}}_{\vec{\mu}}^{\alpha}(K) \quad (44)$$

denote the canonical projection (here  $(D(0; 1)^K \times D(0; 1)^J)^{\circ}$  denotes the open stratum of the diagonal stratification).

Let us define a one dimensional local system  ${}^{\pi}\mathcal{I}_{\vec{\mu}}$  by the same procedure as in 3.1. Its fiber over each positive chamber  $C \in \pi_0((D(0; 1)^K \times D(0; 1)^J)_{\mathbb{R}}^{\circ})$  is identified with  $B$ . Monodromies along the standard paths are given by the formulas

$${}^C T_{ij} = \zeta^{-\pi(i) \cdot \pi(j)}, \quad {}^C T_{ik} = \zeta^{2\mu_k \cdot \pi(i)'}, \quad {}^C T_{km} = \zeta^{-\mu_k \cdot \mu_m}, \quad (45)$$

$i, j \in J$ ,  $i \neq j$ ;  $k, m \in K$ ,  $k \neq m$ . Here  ${}^C T_{ij}$  and  ${}^C T_{km}$  are half-circles, and  ${}^C T_{ik}$  are full circles. This definition essentially coincides with II.12.6, except for an overall sign.

We define a one-dimensional local system  $\mathcal{I}_\mu^\alpha(\mathcal{K})$  over  $\overset{\circ}{\mathcal{A}}_\mu^\alpha(K)$  as

$$\mathcal{I}_\mu^\alpha := (\pi_* \pi^* \mathcal{I}_\mu^\alpha)^{\text{sgn}} \quad (46)$$

where the superscript  $(\bullet)^{\text{sgn}}$  has the same meaning as in 3.1.

For each non-empty subset  $L \subset K$  we can take a part of weights  $\vec{\mu}_L = (\mu_k)_{k \in L}$  and get a local system  $\mathcal{I}_\mu^\alpha(\mathcal{L})$  over  $\overset{\circ}{\mathcal{A}}_\mu^\alpha(L)$ .

For each marked tree  $(\tau, \vec{\alpha}, \vec{\mu})$  with  $K_\tau \subset K$ , we define the local system  $\mathcal{I}_\mu^{\vec{\alpha}}(\tau)$  as the restriction of  $\mathcal{I}_\mu^\alpha(\mathcal{K}_\tau)$  with respect to embedding (41).

**8.2. Factorization.** . The same construction as in 3.2 defines *factorization isomorphisms*

$$\phi_i = \phi_{i;\vec{\mu}}^{\vec{\alpha}}(\tau) : \mathcal{I}_\mu^{\vec{\alpha}}(\tau) \cong \mathcal{I}_{\vec{\mu}_{\leq i}}^{\vec{\alpha}_{\leq i}}(\tau_{\leq i}) \boxtimes \left[ \times \right]_{\parallel \in \mathcal{K}_\gamma} \mathcal{I}_{\vec{\mu}_{\geq \parallel}}^{\vec{\alpha}_{\geq \parallel}}(\tau_{\geq \parallel}) \quad (47)$$

They satisfy the property of

**8.3. Associativity.** For all  $0 < i < j < p$  squares

$$\begin{array}{ccc} & \mathcal{I}(\tau) & \\ \phi_j \swarrow & & \searrow \phi_i \\ \mathcal{I}(\tau_{\leq i}) \boxtimes \left[ \times \right]_{\downarrow \in \mathcal{K}_\downarrow} \mathcal{I}(\tau_{\geq \uparrow}) & & \mathcal{I}(\tau_{\leq j}) \boxtimes \left[ \times \right]_{\parallel \in \mathcal{K}_\gamma} \mathcal{I}(\tau_{\geq \parallel}) \\ & \searrow \phi_i \boxtimes \text{id} & \swarrow \text{id} \boxtimes \phi_j \\ \mathcal{I}(\tau_{\leq i}) \boxtimes \left[ \times \right]_{\parallel \in \mathcal{K}_\gamma} \mathcal{I}(\tau_{\geq \parallel; \leq i}) \boxtimes \left[ \times \right]_{\downarrow \in \mathcal{K}_\downarrow} \mathcal{I}(\tau_{\geq \uparrow}) & & \end{array}$$

commute. (To unburden the notation we have omitted irrelevant indices — they are restored uniquely.)

**8.4.** The collection of local systems  ${}^K\mathcal{I} = \{\mathcal{I}_\mu^\alpha(\mathcal{L}), \mathcal{L} \subset \mathcal{K}\}$ , together with the factorization isomorphisms defined above, will be called *the cohesive local system over  ${}^K\overset{\circ}{\mathcal{A}}$* .

**8.5.** Let us define perverse sheaves

$$\overset{\bullet}{\mathcal{I}}_\mu^{\vec{\alpha}}(\tau) := j_{!*} \mathcal{I}_\mu^{\vec{\alpha}}(\tau) [\dim \overset{\circ}{\mathcal{A}}_\mu^{\vec{\alpha}}(\tau)] \in \mathcal{M}(\overset{\bullet}{\mathcal{A}}_\mu^{\vec{\alpha}}(\tau); \mathcal{S})$$

where  $j : \overset{\circ}{\mathcal{A}}_\mu^{\vec{\alpha}}(\tau) \hookrightarrow \overset{\bullet}{\mathcal{A}}_\mu^{\vec{\alpha}}(\tau)$  denotes the embedding. By functoriality, the factorization isomorphisms (47) induce isomorphisms

$$\phi_i = \phi_{i;\vec{\mu}}^{\vec{\alpha}}(\tau) : \overset{\bullet}{\mathcal{I}}_\mu^{\vec{\alpha}}(\tau) \cong \overset{\bullet}{\mathcal{I}}_{\vec{\mu}_{\leq i}}^{\vec{\alpha}_{\leq i}}(\tau_{\leq i}) \boxtimes \left[ \times \right]_{k \in K_i} \overset{\bullet}{\mathcal{I}}_{\vec{\mu}_{\geq k}}^{\vec{\alpha}_{\geq k}}(\tau_{\geq k}) \quad (48)$$

These isomorphisms satisfy an associativity property completely analogous to 8.3; one should only replace  $\mathcal{I}$  by  $\dot{\mathcal{I}}$  in the diagrams.

## 9. FACTORIZABLE SHEAVES OVER ${}^K\mathcal{A}$

We keep the assumptions of the previous section.

9.1. The first goal of this section is to define a  $B$ -linear category  ${}^K\widetilde{\mathcal{FS}}$  whose objects will be called *factorizable sheaves (over  $({}^K\mathcal{A}, {}^\kappa\mathcal{I})$ )*. Similarly to  $\widetilde{\mathcal{FS}}$ , this category is by definition a product of  $B$ -categories

$${}^K\widetilde{\mathcal{FS}} = \prod_{\vec{c} \in \pi_0(\mathcal{A})^\kappa} {}^K\widetilde{\mathcal{FS}}_{\vec{c}}. \quad (49)$$

Objects of  ${}^K\widetilde{\mathcal{FS}}_{\vec{c}}$  will be called *factorizable sheaves supported at  $\vec{c}$* .

9.2. **Definition.** A factorizable sheaf  $\mathcal{X}$  over  $({}^K\mathcal{A}, {}^\kappa\mathcal{I})$  supported at  $\vec{c} = (c_k) \in \pi_0(\mathcal{A})^\kappa$  is the following collection of data:

(a) a  $K$ -tuple of weights  $\vec{\lambda} = (\lambda_k) \in X^K$  such that  $\lambda_k \in X_{c_k}$ , to be denoted by  $\vec{\lambda}(\mathcal{X})$ ;

(b) for each  $\alpha \in \mathbb{N}[\mathbb{I}]$  a sheaf  $\mathcal{X}^\alpha(\mathcal{K}) \in \mathcal{M}(\mathcal{A}_\lambda^\alpha(\mathcal{K}); \mathcal{S})$ .

Taking restrictions, as in 8.1, we get for each  $K$ -based enhanced tree  $(\tau, \vec{\alpha})$  sheaves  $\mathcal{X}^{\vec{\alpha}}(\tau) \in \mathcal{M}(\mathcal{A}_\lambda^{\vec{\alpha}}(\tau); \mathcal{S})$ .

(c) For each enhanced tree  $(\tau, \vec{\alpha})$  of height 2,  $\tau = (K \xrightarrow{\text{id}} K \longrightarrow \{*\}; (1, d, 0))$ ,  $\vec{\alpha} = (\alpha, \vec{\beta})$  where  $\alpha \in \mathbb{N}[\mathbb{I}]$ ;  $\vec{\beta} \in \mathbb{N}[\mathbb{I}]^\mathbb{K}$ , a factorization isomorphism

$$\psi(\tau) : \mathcal{X}^{(\alpha, \vec{\beta})}(\tau) \cong \dot{\mathcal{I}}_{\vec{\lambda}(\tau) \leq \infty}^\alpha(\tau_{\leq \infty}) \boxtimes \mathcal{X}^{(\iota, \vec{\beta})}(\tau) \quad (50)$$

These isomorphisms should satisfy

Associativity axiom.

For all enhanced trees  $(\tau, \vec{\alpha})$  of height 3,  $\tau = (K \xrightarrow{\text{id}} K \xrightarrow{\text{id}} K \longrightarrow \{*\}; (1, d_1, d_2, 0))$ ,  $\vec{\alpha} = (\alpha, \vec{\beta}, \vec{\gamma})$  where  $\alpha \in \mathbb{N}[\mathbb{I}]$ ;  $\vec{\beta}, \vec{\gamma} \in \mathbb{N}[\mathbb{I}]^\mathbb{K}$ , the square

$$\begin{array}{ccc}
& \mathcal{X}^{(\alpha, \vec{\beta}, \vec{\gamma})}(\tau) & \\
\psi_2 \swarrow & & \searrow \psi_1 \\
\mathcal{I}_{\vec{\lambda}(\tau) \leq 2}^{\bullet(\alpha, \vec{\beta})}(\tau_{\leq 2}) \boxtimes \mathcal{X}^{(\iota, \vec{\gamma})}(\tau) & & \mathcal{I}_{\vec{\lambda}(\tau) \leq 1}^{\bullet\alpha}(\tau_{\leq 1}) \boxtimes \mathcal{X}^{(\iota, \vec{\beta}, \vec{\gamma})}(\tau) \\
\phi_1 \boxtimes \text{id} \searrow & & \swarrow \text{id} \boxtimes \psi_2 \\
\mathcal{I}_{\vec{\lambda}(\tau) \leq 1}^{\bullet\alpha}(\tau_{\leq 1}) \boxtimes \boxed{\times}_{k \in K} & & \mathcal{I}_{\vec{\lambda}(\tau) \geq k; \leq 2}^{\bullet\beta_k}(\tau_{\geq k; \leq 2}) \boxtimes \mathcal{X}^{(\iota, \vec{\gamma})}(\tau)
\end{array}$$

commutes.

9.3. Let  $\mathcal{X}$  be as above. For each  $\vec{\mu} \in X^K$ ,  $\vec{\mu} \geq \vec{\lambda}$ , so that  $\vec{\mu} = \vec{\lambda} + \vec{\beta}'$  for some  $\vec{\beta}' \in \mathbb{N}[\mathbb{I}]^{\mathbb{K}}$ , and  $\alpha \in \mathbb{N}[\mathbb{I}]$ , let us define a sheaf  $\mathcal{X}_{\vec{\mu}}^{\alpha}(\mathcal{K}) \in \mathcal{M}(\mathcal{A}_{\vec{\mu}}^{\alpha}(\mathcal{K}); \mathcal{S})$  as  $\sigma_* \mathcal{X}^{\alpha - \beta}(\mathcal{K})$ . For example,  $\mathcal{X}_{\vec{\lambda}}^{\alpha}(\mathcal{K}) = \mathcal{X}^{\alpha}(\mathcal{K})$ .

Taking restrictions, the sheaves  $\mathcal{X}_{\vec{\mu}}^{\vec{\alpha}}(\tau) \in \mathcal{M}(\mathcal{A}_{\vec{\mu}}^{\vec{\alpha}}(\tau); \mathcal{S})$  for all  $K$ -based trees  $\tau$  are defined.

9.4. Suppose  $\mathcal{X}, \mathcal{Y}$  are two factorizable sheaves supported at  $\vec{c}$ ,  $\vec{\lambda} = \vec{\lambda}(\mathcal{X})$ ,  $\vec{\nu} = \vec{\lambda}(\mathcal{Y})$ . Let  $\vec{\mu} \in X^K$ ,  $\vec{\mu} \geq \vec{\lambda}$ ,  $\vec{\mu} \geq \vec{\nu}$ . By definition, we have canonical isomorphisms

$$\theta = \theta_{\vec{\mu}; \vec{\beta}}^{\alpha} : \text{Hom}_{\mathcal{A}_{\vec{\mu}}^{\alpha}(\mathcal{K})}(\mathcal{X}_{\vec{\mu}}^{\alpha}(\mathcal{K}), \mathcal{Y}_{\vec{\mu}}^{\alpha}(\mathcal{K})) \xrightarrow{\sim} \text{Hom}_{\mathcal{A}_{\vec{\mu} + \vec{\beta}'}^{\alpha + \beta}(\mathcal{K})}(\mathcal{X}_{\vec{\mu} + \vec{\beta}'}^{\alpha + \beta}(\mathcal{K}), \mathcal{Y}_{\vec{\mu} + \vec{\beta}'}^{\alpha + \beta}(\mathcal{K})) \quad (51)$$

for each  $\alpha \in \mathbb{N}[\mathbb{I}]$ ,  $\vec{\beta} \in \mathbb{N}[\mathbb{I}]^{\mathbb{K}}$ .

9.4.1. Suppose we are given  $\vec{\beta} = (\beta_k) \in \mathbb{N}[\mathbb{I}]^{\mathbb{K}}$ . Let  $\beta = \sum_k \beta_k$  as usually. Choose a real  $d$ ,  $0 < d < 1$ .

Consider a marked tree  $(\tau_d, (0, \vec{\beta}), \vec{\mu})$  where

$$\tau_d = (K \xrightarrow{\text{id}} K \longrightarrow \{*\}; (0, d, 1)).$$

We have the restriction homomorphism

$$\xi_{\vec{\mu}; \vec{\beta}; d} : \text{Hom}_{\mathcal{A}_{\vec{\mu}}^{\beta}(\mathcal{K})}(\mathcal{X}_{\vec{\mu}}^{\beta}(\mathcal{K}), \mathcal{Y}_{\vec{\mu}}^{\beta}(\mathcal{K})) \longrightarrow \text{Hom}_{\mathcal{A}_{\vec{\mu}}^{(\iota, \vec{\beta})}(\tau_{\uparrow})}(\mathcal{X}_{\vec{\mu}}^{(\iota, \vec{\beta})}(\tau_{\uparrow}), \mathcal{Y}_{\vec{\mu}}^{(\iota, \vec{\beta})}(\tau_{\uparrow})) \quad (52)$$

Suppose we are given  $'\vec{\beta} = (' \beta_k) \in \mathbb{N}[\mathbb{I}]^{\mathbb{K}}$  such that  $'\vec{\beta} \leq \vec{\beta}$ . Let  $'\beta = \sum_k ' \beta_k$  as usually. Choose a real  $\varepsilon$ ,  $0 < \varepsilon < d$ .

The restriction and the factorization isomorphisms  $\psi$  induce the map

$$\eta_{\vec{\mu}; ' \vec{\beta}; \varepsilon}^{\vec{\beta}; d} : \text{Hom}_{\mathcal{A}_{\vec{\mu}}^{(\iota, \vec{\beta})}(\tau_{\uparrow})}(\mathcal{X}_{\vec{\mu}}^{(\iota, \vec{\beta})}(\tau_{\uparrow}), \mathcal{Y}_{\vec{\mu}}^{(\iota, \vec{\beta})}(\tau_{\uparrow})) \longrightarrow \text{Hom}_{\mathcal{A}_{\vec{\mu}}^{(\iota, ' \vec{\beta})}(\tau_{\varepsilon})}(\mathcal{X}_{\vec{\mu}}^{(\iota, ' \vec{\beta})}(\tau_{\varepsilon}), \mathcal{Y}_{\vec{\mu}}^{(\iota, ' \vec{\beta})}(\tau_{\varepsilon})) \quad (53)$$

The associativity axiom implies that these maps satisfy an obvious transitivity property.

We define the space  $\text{Hom}_{\kappa\widetilde{\mathcal{FS}}}(\mathcal{X}, \mathcal{Y})$  as the following inductive-projective limit

$$\text{Hom}_{\kappa\widetilde{\mathcal{FS}}}(\mathcal{X}, \mathcal{Y}) := \varinjlim \varprojlim \text{Hom}_{\mathcal{A}_{\vec{\mu}}^{\alpha}(\mathcal{K})}(\mathcal{X}_{\vec{\mu}}^{\alpha}(\mathcal{K}), \mathcal{Y}_{\vec{\mu}}^{\alpha}(\mathcal{K})) \quad (54)$$

where the inverse limit is understood as follows. Its elements are collections of maps

$$\{f_K^{\alpha} : \mathcal{X}_{\vec{\mu}}^{\alpha}(\mathcal{K}) \longrightarrow \mathcal{Y}_{\vec{\mu}}^{\alpha}(\mathcal{K})\}$$

given for all  $\alpha \in \mathbb{N}[\mathbb{I}]$ ,  $\vec{\beta} \in \mathbb{N}[\mathbb{I}]^{\mathbb{K}}$ , such that for every  $\alpha$ ,  $'\vec{\beta} \leq \vec{\beta}$ ,  $0 < \varepsilon < d < 1$  as above, we have

$$\eta_{\vec{\mu}; '\vec{\beta}; \varepsilon}^{\vec{\beta}; d} \xi_{\vec{\mu}; \vec{\beta}; d}(f_K^{\beta}) = \xi_{\vec{\mu}; '\vec{\beta}; \varepsilon}(f_K^{'\beta})$$

$\vec{\mu}$  being fixed. The direct limit is taken over  $\vec{\mu} \in X^K$  such that  $\vec{\mu} \geq \vec{\lambda}$ ,  $\vec{\mu} \geq \vec{\nu}$ , the transition maps being induced by (51).

With these spaces of homomorphisms, factorizable sheaves supported at  $\vec{c}$  form a  $B$ -linear category to be denoted by  ${}^K\widetilde{\mathcal{FS}}_{\vec{c}}$ . As we have already mentioned, the category of factorizable sheaves  ${}^K\widetilde{\mathcal{FS}}$  is by definition the product (49).

## FINITE SHEAVES

**9.5. Definition.** A sheaf  $\mathcal{X} \in {}^{\kappa}\widetilde{\mathcal{FS}}_{\vec{c}}$  is called *finite* if there exists only finitely many  $\vec{\beta} \in \mathbb{N}[\mathbb{I}]^{\mathbb{K}}$  such that the singular support of  $\mathcal{X}_{\vec{\lambda}}^{\alpha}(\mathcal{K})$  contains the conormal bundle to  $\sigma_{\vec{\lambda}-\vec{\beta}; \vec{\beta}}^{\alpha-\beta}(\mathcal{A}_{\vec{\lambda}-\vec{\beta}}^{\alpha-\beta})$  (see (43)) for  $\alpha \geq \beta = \sum_k \beta_k$ .

A sheaf  $\mathcal{X} = \bigoplus_{\vec{c}} \mathcal{X}_{\vec{c}} \in {}^{\kappa}\widetilde{\mathcal{FS}}$ ,  $\mathcal{X}_{\vec{c}} \in {}^{\kappa}\widetilde{\mathcal{FS}}_{\vec{c}}$  is called *finite* if all  $\mathcal{X}_{\vec{c}}$  are finite.

**9.6.** Suppose we are given finite sheaves  $\mathcal{X}, \mathcal{Y} \in {}^{\kappa}\mathcal{FS}_{\vec{c}}$ ; and  $\vec{\mu} \geq \vec{\lambda}(\mathcal{X}), \vec{\lambda}(\mathcal{Y})$ . As in the proof of the Lemma 5.3, one can see that there exists  $'\vec{\beta} \in \mathbb{N}[\mathbb{I}]^{\mathbb{K}}$  such that for any  $\vec{\beta} \geq '\vec{\beta}$  the map

$$\eta_{\vec{\mu}; \vec{\beta}; \varepsilon}^{'\vec{\beta}; d} : \text{Hom}_{\mathcal{A}_{\vec{\mu}}^{(\iota, \vec{\beta})}(\tau_{\vec{c}})}(\mathcal{X}_{\vec{\mu}}^{(\iota, \vec{\beta})}(\tau_{\vec{c}}), \mathcal{Y}_{\vec{\mu}}^{(\iota, \vec{\beta})}(\tau_{\vec{c}})) \longrightarrow \text{Hom}_{\mathcal{A}_{\vec{\mu}}^{(\iota, '\vec{\beta})}(\tau_{\varepsilon})}(\mathcal{X}_{\vec{\mu}}^{(\iota, '\vec{\beta})}(\tau_{\varepsilon}), \mathcal{Y}_{\vec{\mu}}^{(\iota, '\vec{\beta})}(\tau_{\varepsilon})) \quad (55)$$

is an isomorphism. We will identify all the spaces  $\text{Hom}_{\mathcal{A}_{\vec{\mu}}^{(\iota, \vec{\beta})}(\tau_{\vec{c}})}(\mathcal{X}_{\vec{\mu}}^{(\iota, \vec{\beta})}(\tau_{\vec{c}}), \mathcal{Y}_{\vec{\mu}}^{(\iota, \vec{\beta})}(\tau_{\vec{c}}))$  with the help of the above isomorphisms, and we will denote this stabilized space by  $\overline{\text{Hom}}_{\kappa\mathcal{FS}}(\mathcal{X}, \mathcal{Y})$ . Evidently, it does not depend on a choice of  $'\vec{\beta}$ .

Quite similarly to the *loc.cit* one can see that for any  $\vec{\beta} \geq '\vec{\beta}$  the map

$$\xi_{\vec{\mu}; \vec{\beta}; d} : \text{Hom}_{\mathcal{A}_{\vec{\mu}}^{\beta}(\mathcal{K})}(\mathcal{X}_{\vec{\mu}}^{\beta}(\mathcal{K}), \mathcal{Y}_{\vec{\mu}}^{\beta}(\mathcal{K})) \longrightarrow \text{Hom}_{\mathcal{A}_{\vec{\mu}}^{(\iota, \vec{\beta})}(\tau_{\vec{c}})}(\mathcal{X}_{\vec{\mu}}^{(\iota, \vec{\beta})}(\tau_{\vec{c}}), \mathcal{Y}_{\vec{\mu}}^{(\iota, \vec{\beta})}(\tau_{\vec{c}})) \quad (56)$$

is an injection.

Thus we may view  $\mathrm{Hom}_{\mathcal{A}_\mu^\beta(\mathcal{K})}(\mathcal{X}_\mu^\beta(\mathcal{K}), \mathcal{Y}_\mu^\beta(\mathcal{K}))$  as the subspace of  $\overline{\mathrm{Hom}}_{\kappa\mathcal{FS}}(\mathcal{X}, \mathcal{Y})$ .

We define  $\mathrm{Hom}_{\kappa\mathcal{FS}}(\mathcal{X}, \mathcal{Y}) \subset \overline{\mathrm{Hom}}_{\kappa\mathcal{FS}}(\mathcal{X}, \mathcal{Y})$  as the projective limit of the system of subspaces  $\mathrm{Hom}_{\mathcal{A}_\mu^\beta(\mathcal{K})}(\mathcal{X}_\mu^\beta(\mathcal{K}), \mathcal{Y}_\mu^\beta(\mathcal{K}))$ ,  $\vec{\beta} \geq \vec{\beta}'$ .

With such definition of morphisms finite factorizable sheaves supported at  $\vec{c}$  form an abelian category to be denoted by  ${}^K\mathcal{FS}_{\vec{c}}$ . We set by definition

$${}^K\mathcal{FS} = \prod_{\vec{c} \in \pi_!(\mathcal{A})^\mathcal{K}} {}^K\mathcal{FS}_{\vec{c}} \quad (57)$$

## 10. GLUING

10.1. Let

$$\mathcal{A}_{\mu;\infty}^\alpha \subset \mathcal{A}_\mu^\alpha$$

denote an open configuration subspace parametrizing configurations lying entirely inside the unit disk  $D(0;1)$ . Due to monodromicity, the restriction functors

$$\mathcal{M}(\mathcal{A}_\mu^\alpha; \mathcal{S}) \longrightarrow \mathcal{M}(\mathcal{A}_{\mu;\infty}^\alpha; \mathcal{S})$$

are equivalences.

Let  $\{*\}$  denote a one-element set. We have closed embeddings

$$i : \mathcal{A}_{\mu;\infty}^\alpha \hookrightarrow \mathcal{A}_\mu^\alpha(\{*\}),$$

which identify the first space with the subspace of the second one consisting of configurations with the small disk centered at 0. The inverse image functors

$$i^*[-1] : \mathcal{M}(\mathcal{A}_\mu^\alpha(\{*\}); \mathcal{S}) \longrightarrow \mathcal{M}(\mathcal{A}_{\mu;\infty}^\alpha; \mathcal{S}) \quad (58)$$

are equivalences, again due to monodromicity. Thus, we get equivalences

$$\mathcal{M}(\mathcal{A}_\mu^\alpha; \mathcal{S}) \xrightarrow{\sim} \mathcal{M}(\mathcal{A}_\mu^\alpha(\{*\}); \mathcal{S})$$

which induce canonical equivalences

$$\widetilde{\mathcal{FS}} \xrightarrow{\sim} \widetilde{\mathcal{FS}}^{\{*\}} \quad (59)$$

and

$$\mathcal{FS} \xrightarrow{\sim} \mathcal{FS}^{\{*\}} \quad (60)$$

Using these equivalences, we will sometimes identify these categories.

**10.2. Tensor product of categories.** Let  $\mathcal{B}_\infty, \mathcal{B}_\infty$  be  $B$ -linear abelian categories. Their tensor product category  $\mathcal{B}_\infty \otimes \mathcal{B}_\infty$  is defined in §5 of [D2]. It comes together with a canonical right biexact functor  $\mathcal{B}_\infty \times \mathcal{B}_\infty \longrightarrow \mathcal{B}_\infty \otimes \mathcal{B}_\infty$ , and it is the initial object among such categories.



10.2.1. *Basic Example.* Let  $M_i$ ,  $i = 1, 2$ , be complex algebraic varieties equipped with algebraic Whitney stratifications  $\mathcal{S}_i$ . Let  $\mathcal{B}_\gamma = \mathcal{M}(\mathcal{M}_\gamma; \mathcal{S}_\gamma)$ . Then

$$\mathcal{B}_\infty \otimes \mathcal{B}_\epsilon = \mathcal{M}(\mathcal{M}_\infty \times \mathcal{M}_\epsilon; \mathcal{S}_\infty \times \mathcal{S}_\epsilon).$$

The canonical functor  $\mathcal{B}_\infty \times \mathcal{B}_\epsilon \longrightarrow \mathcal{B}_\infty \otimes \mathcal{B}_\epsilon$  sends  $(\mathcal{X}_\infty, \mathcal{X}_\epsilon)$  to  $\mathcal{X}_\infty \boxtimes \mathcal{X}_\epsilon$ .

10.2.2. Recall the notations of 9.4.1. Let us consider the following category  $\mathcal{FS}^{\otimes \mathcal{K}}$ . Its objects are the collections of perverse sheaves  $\mathcal{X}_\mu^{(\iota, \vec{\beta})}(\tau_\Gamma)$  on the spaces  $\mathcal{A}_\mu^{(\iota, \vec{\beta})}(\tau_\Gamma)$  for sufficiently small  $d$ , satisfying the usual factorization and finiteness conditions. The morphisms are defined via the inductive-projective system with connecting maps  $\eta_{\vec{\mu}; \vec{\beta}; \epsilon}^{\vec{\beta}; d}$ . Using the above Basic Example, one can see easily that the category  $\mathcal{FS}^{\otimes \mathcal{K}}$  is canonically equivalent to  $\mathcal{FS} \otimes \dots \otimes \mathcal{FS}$  ( $K$  times) which justifies its name.

By definition, the category  ${}^K\mathcal{FS}$  comes together with the functor  $p_K : {}^K\mathcal{FS} \longrightarrow \mathcal{FS}^{\otimes \mathcal{K}}$  injective on morphisms. In effect,

$$\mathrm{Hom}_{{}^K\mathcal{FS}}(\mathcal{X}, \mathcal{Y}) \hookrightarrow \overline{\mathrm{Hom}}_{{}^K\mathcal{FS}}(\mathcal{X}, \mathcal{Y}) = \mathrm{Hom}_{\mathcal{FS}^{\otimes \mathcal{K}}}(\sqrt{\mathcal{K}}(\mathcal{X}), \sqrt{\mathcal{K}}(\mathcal{Y})).$$

Let us construct a functor in the opposite direction.

10.3. **Gluing of factorizable sheaves.** For each  $0 < d < 1$  let us consider a tree

$$\tau_d = (K \xrightarrow{\mathrm{id}} K \longrightarrow \{*\}; (1, d, 0)).$$

Suppose we are given  $\alpha \in \mathbb{N}[\mathbb{I}]$ . Let  $\mathcal{V}(\alpha)$  denote the set of all enhancements  $\vec{\alpha} = (\alpha_*; (\alpha_k)_{k \in K})$  of  $\tau$  such that  $\alpha_* + \sum_{k \in K} \alpha_k = \alpha$ . Obviously, the open subspaces  $\mathcal{A}^{\vec{\alpha}}(\tau_\Gamma) \subset \mathcal{A}^\alpha(\mathcal{K})$ , for varying  $d$  and  $\vec{\alpha} \in \mathcal{V}(\alpha)$ , form an open covering of  $\mathcal{A}^\alpha(\mathcal{K})$ .

Suppose we are given a collection of factorizable sheaves  $\mathcal{X}_\parallel \in \mathcal{FS}_\parallel$ ,  $\parallel \in \mathcal{K}$ . Set  $\vec{\lambda} = (\lambda(\mathcal{X}_\parallel)) \in \mathcal{X}^\mathcal{K}$ . For each  $d, \vec{\alpha}$  as above consider a sheaf

$$\mathcal{X}^{\vec{\alpha}}(\tau_\Gamma) := \mathcal{I}_{\vec{\lambda}_{\leq \infty}}^{\bullet, \alpha_*}(\tau_{\Gamma; \leq \infty}) \boxtimes \boxed{\times}_{\parallel \in \mathcal{K}} \mathcal{X}_\parallel^{\alpha_\parallel}$$

over  $\mathcal{A}_\lambda^{\vec{\alpha}}(\tau_\Gamma)$ .

Non-trivial pairwise intersections of the above open subspaces look as follows. For  $0 < d_2 < d_1 < 1$ , consider a tree of height 3

$$\varsigma = \varsigma_{d_1, d_2} = (K \xrightarrow{\mathrm{id}} K \xrightarrow{\mathrm{id}} K \longrightarrow \{*\}; (1, d_1, d_2, 0)).$$

We have  $\partial_1 \varsigma = \tau_{d_2}$ ,  $\partial_2 \varsigma = \tau_{d_1}$ . Let  $\vec{\beta} = (\beta_*, (\beta_{1;k})_{k \in K}, (\beta_{2;k})_{k \in K})$  be an enhancement of  $\varsigma$ . Set  $\vec{\alpha}_1 = \partial_2 \vec{\beta}$ ,  $\vec{\alpha}_2 = \partial_1 \vec{\beta}$ . Note that  $\vec{\beta}$  is defined uniquely by  $\vec{\alpha}_1, \vec{\alpha}_2$ . We have

$$\mathcal{A}_\lambda^{\vec{\beta}}(\varsigma) = \mathcal{A}_\lambda^{\vec{\alpha}_1 \infty}(\tau_{\Gamma_\infty}) \cap \mathcal{A}_\lambda^{\vec{\alpha}_2 \in}(\tau_{\Gamma_\epsilon}).$$

Due to the factorization property for sheaves  $\dot{\mathcal{I}}$  and  $\mathcal{X}_{\parallel}$  we have isomorphisms

$$\mathcal{X}^{\vec{\alpha}\infty}(\tau_{\top\infty})|_{\mathcal{A}_{\vec{\lambda}}^{\vec{\beta}(\varsigma)}} \cong \dot{\mathcal{I}}_{\partial_{\vec{\lambda}}\vec{\lambda}}^{\partial_{\vec{\lambda}}\vec{\beta}}(\partial_{\vec{\lambda}}\varsigma) \boxtimes \boxed{\times}_{\parallel \in \mathcal{K}}(\mathcal{X}_{\parallel})_{\lambda_{\parallel}}^{\beta_{\in\parallel}},$$

and

$$\mathcal{X}^{\vec{\alpha}\in}(\tau_{\top\in})|_{\mathcal{A}_{\vec{\lambda}}^{\vec{\beta}(\varsigma)}} \cong \dot{\mathcal{I}}_{\partial_{\vec{\lambda}}\vec{\lambda}}^{\partial_{\vec{\lambda}}\vec{\beta}}(\partial_{\vec{\lambda}}\varsigma) \boxtimes \boxed{\times}_{\parallel \in \mathcal{K}}(\mathcal{X}_{\parallel})_{\lambda_{\parallel}}^{\beta_{\in\parallel}}$$

Taking composition, we get isomorphisms

$$\phi_{d_1, d_2}^{\vec{\alpha}_1, \vec{\alpha}_2} : \mathcal{X}^{\vec{\alpha}\infty}(\tau_{\top\infty})|_{\mathcal{A}_{\vec{\lambda}}^{\vec{\alpha}\infty}(\tau_{\top\infty}) \cap \mathcal{A}_{\vec{\lambda}}^{\vec{\alpha}\in}(\tau_{\top\in})} \xrightarrow{\sim} \mathcal{X}^{\vec{\alpha}\in}(\tau_{\top\in})|_{\mathcal{A}_{\vec{\lambda}}^{\vec{\alpha}\infty}(\tau_{\top\infty}) \cap \mathcal{A}_{\vec{\lambda}}^{\vec{\alpha}\in}(\tau_{\top\in})} \quad (61)$$

From the associativity of the factorization for the sheaves  $\dot{\mathcal{I}}$  and  $\mathcal{X}_{\parallel}$  it follows that the isomorphisms (61) satisfy the cocycle condition; hence they define a sheaf  $\mathcal{X}^{\alpha}(\mathcal{K})$  over  $\mathcal{A}_{\vec{\lambda}}^{\alpha}(\mathcal{K})$ .

Thus, we have defined a collection of sheaves  $\{\mathcal{X}^{\alpha}(\mathcal{K})\}$ . Using the corresponding data for the sheaves  $\mathcal{X}_{\parallel}$ , one defines easily factorization isomorphisms 9.2 (d) and check that they satisfy the associativity property. One also sees immediately that the collection of sheaves  $\{\mathcal{X}^{\alpha}(\mathcal{K})\}$  is finite. We leave this verification to the reader.

This way we get maps

$$\prod_k \text{Ob}(\mathcal{FS}_{\parallel}) \longrightarrow \text{Ob}({}^{\mathcal{K}}\mathcal{FS}_{\vec{\top}}), \quad \vec{\top} = (\parallel)$$

which extend by additivity to the map

$$g_K : \text{Ob}(\mathcal{FS}^{\mathcal{K}}) \longrightarrow \text{Ob}({}^{\mathcal{K}}\mathcal{FS}) \quad (62)$$

To construct the functor

$$g_K : \mathcal{FS}^{\mathcal{K}} \longrightarrow {}^{\mathcal{K}}\mathcal{FS} \quad (63)$$

it remains to define  $g_K$  on morphisms.

Given two collections of finite factorizable sheaves  $\mathcal{X}_{\parallel}, \mathcal{Y}_{\parallel} \in \mathcal{FS}_{\parallel}$ ,  $\parallel \in \mathcal{K}$ , let us choose  $\vec{\lambda} = (\lambda_k)_{k \in K}$  such that  $\lambda_k \geq \lambda(\mathcal{X}_{\parallel}), \lambda(\mathcal{Y}_{\parallel})$  for all  $k \in K$ . Suppose we have a collection of morphisms  $f_k : \mathcal{X}_{\parallel} \longrightarrow \mathcal{Y}_{\parallel}$ ,  $\parallel \in \mathcal{K}$ ; that is the maps  $f_k^{\alpha_k} : (\mathcal{X}_{\parallel})_{\lambda_{\parallel}}^{\alpha_{\parallel}} \longrightarrow (\mathcal{Y}_{\parallel})_{\lambda_{\parallel}}^{\alpha_{\parallel}}$  given for any  $\alpha_k \in \mathbb{N}[\mathbb{I}]$  compatible with factorizations.

Given  $\alpha \in \mathbb{N}[\mathbb{I}]$  and an enhancement  $\vec{\alpha} \in \mathcal{V}(\alpha)$  as above we define the morphism  $f^{\vec{\alpha}}(\tau_d) : \mathcal{X}^{\vec{\alpha}}(\tau_{\top}) \longrightarrow \mathcal{Y}^{\vec{\alpha}}(\tau_{\top})$  over  $\mathcal{A}_{\vec{\lambda}}^{\vec{\alpha}}(\tau_{\top})$  as follows: it is the tensor product of the identity on  $\dot{\mathcal{I}}_{\vec{\lambda}_{\leq 1}}^{\alpha_{*}}(\tau_{d; \leq 1})$  with the morphisms  $f_k^{\alpha_k} : (\mathcal{X}_{\parallel})_{\lambda_{\parallel}}^{\alpha_{\parallel}} \longrightarrow (\mathcal{Y}_{\parallel})_{\lambda_{\parallel}}^{\alpha_{\parallel}}$ .

One sees easily as above that the morphisms  $f^{\vec{\alpha}}(\tau_d)$  glue together to give a morphism  $f^{\alpha}(K) : \mathcal{X}^{\alpha}(\mathcal{K}) \longrightarrow \mathcal{Y}^{\alpha}(\mathcal{K})$ ; as  $\alpha$  varies they provide a morphism  $f(K) : g_K((\mathcal{X}_{\parallel})) \longrightarrow$

$\}_{\mathcal{K}}((\mathcal{Y}_{\parallel}))$ . Thus we have defined the desired functor  $g_K$ . Obviously it is  $K$ -exact, so by universal property it defines the same named functor

$$g_K : \mathcal{FS}^{\otimes \mathcal{K}} \longrightarrow {}^K\mathcal{FS} \quad (64)$$

By the construction, the composition  $p_K \circ g_K : \mathcal{FS}^{\otimes \mathcal{K}} \longrightarrow \mathcal{FS}^{\otimes \mathcal{K}}$  is isomorphic to the identity functor. Recalling that  $p_K$  is injective on morphisms we see that  $g_K$  and  $p_K$  are quasiinverse. Thus we get

**10.4. Theorem.** *The functors  $p_K$  and  $g_K$  establish a canonical equivalence*

$${}^K\mathcal{FS} \xrightarrow{\sim} \mathcal{FS}^{\otimes \mathcal{K}} \quad \square$$

## 11. FUSION

### BRAIDED TENSOR CATEGORIES

In this part we review the definition of a braided tensor category following Deligne, [D1].

**11.1.** Let  $\mathcal{C}$  be a category,  $Y$  a locally connected locally simply connected topological space. By a *local system* over  $Y$  with values in  $\mathcal{C}$  we will mean a locally constant sheaf over  $Y$  with values in  $\mathcal{C}$ . They form a category to be denoted by  $\mathcal{L}|f\uparrow f(\mathcal{Y}; \mathcal{C})$ .

**11.1.1.** We will use the following basic example. If  $X$  is a complex algebraic variety with a Whitney stratification  $\mathcal{S}$  then the category  $\mathcal{M}(\mathcal{X} \times \mathcal{Y}; \mathcal{S} \times \mathcal{S}_{Y;\sqcup \nabla})$  is equivalent to  $\mathcal{L}|f\uparrow f(\mathcal{Y}; \mathcal{M}(\mathcal{X}; \mathcal{S}))$ . Here  $S_{Y;tr}$  denotes the trivial stratification of  $Y$ , i.e. the first category consists of sheaves smooth along  $Y$ .

**11.2.** Let  $\pi : K \longrightarrow L$  be an epimorphism of non-empty finite sets. We will use the notations of 7.3. For real  $\epsilon, \delta$  such that  $1 > \epsilon > \delta > 0$ , consider a tree

$$\tau_{\pi; \epsilon, \delta} = (K \xrightarrow{\pi} L \longrightarrow \{*\}; (1, \epsilon, \delta))$$

We have an isomorphism which is a particular case of (36):

$$\mathcal{O}(\tau_{\pi; \epsilon, \delta}) \cong \mathcal{O}(\mathcal{L}; \epsilon) \times \prod_{\uparrow \in \mathcal{L}} \mathcal{O}(\mathcal{K}_{\uparrow}; \delta \epsilon^{-\infty})$$

where  $K_l := \pi^{-1}(l)$ .

11.2.1. **Lemma** *There exists essentially unique functor*

$$r_\pi : \mathcal{L}\mathcal{I}f\mathcal{I}f(\mathcal{O}(\mathcal{K}); \mathcal{C}) \longrightarrow \mathcal{L}\mathcal{I}f\mathcal{I}f(\mathcal{O}(\mathcal{L}) \times \prod_{\downarrow} \mathcal{O}(\mathcal{K}_{\downarrow}); \mathcal{C})$$

such that for each  $\epsilon, \delta$  as above the square

$$\begin{array}{ccc} \mathcal{L}\mathcal{I}f\mathcal{I}f(\mathcal{O}(\mathcal{K}); \mathcal{C}) & \xrightarrow{r_\pi} & \mathcal{L}\mathcal{I}f\mathcal{I}f(\mathcal{O}(\mathcal{L}) \times \prod_{\downarrow} \mathcal{O}(\mathcal{K}_{\downarrow}); \mathcal{C}) \\ \downarrow & & \downarrow \\ \mathcal{L}\mathcal{I}f\mathcal{I}f(\tau_{\pi; \epsilon, \delta}; \mathcal{C}) & \xrightarrow{\sim} & \mathcal{L}\mathcal{I}f\mathcal{I}f(\mathcal{O}(\mathcal{L}; \epsilon) \times \prod_{\downarrow} \mathcal{O}(\mathcal{K}_{\downarrow}; \delta\epsilon^{-\infty}); \mathcal{C}) \end{array}$$

commutes.

**Proof** follows from the remark that  $\mathcal{O}(\mathcal{L})$  is a union of its open subspaces

$$\mathcal{O}(\mathcal{L}) = \bigcup_{\epsilon > 0} \mathcal{O}(\mathcal{L}; \epsilon). \quad \square$$

11.3. Let  $\mathcal{C}$  be a category. A *braided tensor structure* on  $\mathcal{C}$  is the following collection of data.

(i) For each non-empty finite set  $K$  a functor

$$\otimes_K : \mathcal{C}^K \longrightarrow \mathcal{L}\mathcal{I}f\mathcal{I}f(\mathcal{O}(\mathcal{K}); \mathcal{C}), \quad \{\mathcal{X}_{\parallel}\} \mapsto \otimes_K \mathcal{X}_{\parallel}$$

from the  $K$ -th power of  $\mathcal{C}$  to the category of local systems (locally constant sheaves) over the space  $\mathcal{O}(\mathcal{K})$  with values in  $\mathcal{C}$  (we are using the notations of 7.3).

We suppose that  $\otimes_{\{*\}} X$  is the constant local system with the fiber  $X$ .

(ii) For each  $\pi : K \longrightarrow L$  as above a natural isomorphism

$$\phi_\pi : (\otimes_K X_k)|_{\mathcal{O}(\mathcal{L}) \times \prod \mathcal{O}(\mathcal{K}_{\downarrow})} \xrightarrow{\sim} \otimes_L (\otimes_{K_l} X_k).$$

To simplify the notation, we will write this isomorphism in the form

$$\phi_\pi : \otimes_K X_k \xrightarrow{\sim} \otimes_L (\otimes_{K_l} X_k),$$

implying that in the left hand side we must take restriction.

These isomorphisms must satisfy the following

*Associativity axiom.* For each pair of epimorphisms  $K \xrightarrow{\pi} L \xrightarrow{\rho} M$  the square

$$\begin{array}{ccccc}
& & \otimes_K X_k & & \\
& \swarrow \phi_{\rho\pi} & & \searrow \phi_\pi & \\
\otimes_M(\otimes_{K_m} X_k) & & & & \otimes_L(\otimes_{K_l} X_k) \\
& \searrow \otimes_M \phi_{\pi|_{K_m}} & & \swarrow \phi_\rho & \\
& & \otimes_M(\otimes_{L_m}(\otimes_{K_l} X_k)) & & 
\end{array}$$

where  $K_m := (\rho\pi)^{-1}(m)$ ,  $L_m := \rho^{-1}(m)$ , commutes.

11.4. The connection with the conventional definition is as follows. Given two objects  $X_1, X_2 \in \text{Ob } \mathcal{C}$ , define an object  $X_1 \dot{\otimes} X_2$  as the fiber of  $\otimes_{\{1,2\}} X_k$  at the point  $(1/3, 1/2)$ .

We have natural isomorphisms

$$A_{X_1, X_2, X_3} : X_1 \dot{\otimes} (X_2 \dot{\otimes} X_3) \xrightarrow{\sim} (X_1 \dot{\otimes} X_2) \dot{\otimes} X_3$$

coming from isomorphisms  $\phi$  associated with two possible order preserving epimorphic maps  $\{1, 2, 3\} \longrightarrow \{1, 2\}$ , and

$$R_{X_1, X_2} : X_1 \dot{\otimes} X_2 \xrightarrow{\sim} X_2 \dot{\otimes} X_1$$

coming from the standard half-circle monodromy. Associativity axiom for  $\phi$  is equivalent to the the usual compatibilities for these maps.

11.5. Now suppose that the data 11.3 is given for *all* (possibly empty) tuples and all (not necessarily epimorphic) maps. The space  $\mathcal{O}(\emptyset)$  is by definition one point, and a local system  $\otimes_\emptyset$  over it is simply an object of  $\mathcal{C}$ ; let us denote it  $\mathbf{1}$  and call a *unit* of our tensor structure. In this case we will say that  $\mathcal{C}$  is a braided tensor category with unit.

In the conventional language, we have natural isomorphisms

$$\mathbf{1} \dot{\otimes} X \xrightarrow{\sim} X$$

(they correspond to  $\{2\} \hookrightarrow \{1, 2\}$ ) satisfying the usual compatibilities with  $A$  and  $R$ .

## FUSION FUNCTORS

11.6. Let

$$\mathcal{A}_{\alpha; \infty} \subset \mathcal{A}_\alpha$$

denote the open subspace parametrizing configurations lying inside the unit disk  $D(0; 1)$ .

Let  $K$  be a non-empty finite set. Obviously,  $\mathcal{A}^\alpha(K) = \mathcal{A}_{\alpha;\infty} \times \mathcal{O}(K)$ , and we have the projection

$$\mathcal{A}^\alpha(K) \longrightarrow \mathcal{O}(K).$$

Note also that we have an evident open embedding  $\mathcal{O}(K) \hookrightarrow \mathcal{D}(t; \infty)^K$ .

Our aim in this part is to define certain *fusion functors*

$$\Psi_K : \mathcal{D}(\mathcal{A}^\alpha(K)) \longrightarrow \mathcal{D}^{\mathfrak{H}^\lambda}(\mathcal{A}^\alpha(\{*\}) \times \mathcal{O}(K))$$

where  $(\bullet)^{mon}$  denotes the full subcategory of complexes smooth along  $\mathcal{O}(K)$ . The construction follows the classical definition of nearby cycles functor, [D3].

**11.7. Poincaré groupoid.** We start with a topological notation. Let  $X$  be a connected locally simply connected topological space. Let us denote by  $\widetilde{X \times X}$  the space whose points are triples  $(x, y, p)$ , where  $x, y \in X$ ;  $p$  is a homotopy class of paths in  $X$  connecting  $x$  with  $y$ . Let

$$c_X : \widetilde{X \times X} \longrightarrow X \times X \tag{65}$$

be the evident projection. Note that for a fixed  $x \in X$ , the restriction of  $c_X$  to  $c_X^{-1}(X \times \{x\})$  is a universal covering of  $X$  with a group  $\pi_1(X; x)$ .

11.8. Consider the diagram with cartesian squares

$$\begin{array}{ccccccc} \mathcal{A}^\alpha(\{*\}) \times \mathcal{O}(K) & \xrightarrow{\tilde{\Delta}} & \widehat{\mathcal{A}_\alpha(K)} \times \mathcal{O}(K) & \xleftarrow{\tilde{j}} & \mathcal{A}^\alpha(K) \times \mathcal{O}(K) & \xleftarrow{\tilde{c}} & \mathcal{A}^\alpha(K) \times \mathcal{O}(K) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ D(0, 1) \times \mathcal{O}(K) & \xrightarrow{\Delta} & D(0; 1)^K \times \mathcal{O}(K) & \xleftarrow{j} & \mathcal{O}(K) \times \mathcal{O}(K) & \xleftarrow{c} & \mathcal{O}(K) \times \mathcal{O}(K) \end{array}$$

where we have denoted  $\widehat{\mathcal{A}_\alpha(K)} := \mathcal{A}_{\alpha;\infty} \times \mathcal{D}(t; \infty)^K$ . Here  $\Delta$  is induced by the diagonal embedding  $D(0; 1) \hookrightarrow D(0; 1)^K$ ,  $j$  — by the open embedding  $\mathcal{O}(K) \hookrightarrow \mathcal{D}(t; \infty)^K$ ,  $c$  is the map (65). The upper horizontal arrows are defined by pull-back.

We define  $\Psi_K$  as a composition

$$\Psi_K = \tilde{\Delta}^* \tilde{j}_* \tilde{c}_* \tilde{c}^* p^*[1]$$

where  $p : \mathcal{A}^\alpha(K) \times \mathcal{O}(K) \longrightarrow \mathcal{A}^\alpha(K)$  is the projection.

This functor is  $t$ -exact and induces an exact functor

$$\Psi_K : \mathcal{M}(\mathcal{A}^\alpha(K); \mathcal{S}) \longrightarrow \mathcal{M}(\mathcal{A}^\alpha(\{*\}) \times \mathcal{O}(K); \mathcal{S} \times \mathcal{S}_{\square \nabla}) \tag{66}$$

where  $\mathcal{S}_{\sqcup \nabla}$  denotes the trivial stratification of  $\mathcal{O}(\mathcal{K})$ .

11.9. Set

$$\mathcal{A}^\alpha(\mathcal{K})_{\ulcorner} := \mathcal{A}_{\alpha;\infty} \times \mathcal{O}(\mathcal{K}; \ulcorner)$$

The category  $\mathcal{M}(\mathcal{A}^\alpha(\mathcal{K}); \mathcal{S})$  is equivalent to the "inverse limit"  $\varprojlim \mathcal{M}(\mathcal{A}^\alpha(\mathcal{K})_{\ulcorner}; \mathcal{S})$ .

Let  $\pi : K \longrightarrow L$  be an epimorphism. Consider a configuration space

$$\mathcal{A}^\alpha(\tau_{\pi;\ulcorner}) := \mathcal{A}_{\alpha;\infty} \times \mathcal{O}(\tau_{\pi;\ulcorner})$$

where  $\tau_{\pi;d} := \tau_{\pi;d,0}$ . An easy generalization of the definition of  $\Psi_K$  yields a functor

$$\Psi_{\pi;d} : \mathcal{M}(\mathcal{A}^\alpha(\tau_{\pi;\ulcorner})) \longrightarrow \mathcal{M}(\mathcal{A}^\alpha(\mathcal{L})_{\ulcorner} \times \prod_{\uparrow \in \mathcal{L}} \mathcal{O}(\mathcal{K}_{\uparrow}))$$

(In what follows we will omit for brevity stratifications from the notations of abelian categories  $\mathcal{M}(\bullet)$ , implying that we use the principal stratification on all configuration spaces  $\mathcal{A}^\alpha(\bullet)$  and the trivial stratification on spaces  $\mathcal{O}(\bullet)$ , i.e. our sheaves are smooth along these spaces.) Passing to the limit over  $d > 0$  we conclude that there exists essentially unique functor

$$\Psi_{K \longrightarrow L} : \mathcal{M}(\mathcal{A}^\alpha(\mathcal{K})) \longrightarrow \mathcal{M}(\mathcal{A}^\alpha(\mathcal{L})_{\ulcorner} \times \prod_{\uparrow} \mathcal{O}(\mathcal{K}_{\uparrow})) \quad (67)$$

such that all squares

$$\begin{array}{ccc} \mathcal{M}(\mathcal{A}^\alpha(\mathcal{K})) & \xrightarrow{\Psi_{K \longrightarrow L}} & \mathcal{M}(\mathcal{A}^\alpha(\mathcal{L})_{\ulcorner} \times \prod_{\uparrow} \mathcal{O}(\mathcal{K}_{\uparrow})) \\ \downarrow & & \downarrow \\ \mathcal{M}(\mathcal{A}^\alpha(\tau_{\pi;\ulcorner})) & \xrightarrow{\Psi_{\pi;d}} & \mathcal{M}(\mathcal{A}^\alpha(\mathcal{L})_{\ulcorner} \times \prod_{\uparrow} \mathcal{O}(\mathcal{K}_{\uparrow})) \end{array}$$

commute (the vertical arrows being restrictions). If  $L = \{*\}$ , we return to  $\Psi_K$ .

11.10. **Lemma.** *All squares*

$$\begin{array}{ccc}
\mathcal{M}(\mathcal{A}^\alpha(\mathcal{K})) & \xrightarrow{\Psi_K} & \mathcal{M}(\mathcal{A}^\alpha(\{*\}) \times \mathcal{O}(\mathcal{K})) \\
\downarrow \Psi_{K \rightarrow L} & & \downarrow r_\pi \\
\mathcal{M}(\mathcal{A}^\alpha(\mathcal{L}) \times \prod_{\dagger} \mathcal{O}(\mathcal{K}_{\dagger})) & \xrightarrow{\Psi_L} & \mathcal{M}(\mathcal{A}^\alpha(\{*\}) \times \mathcal{O}(\mathcal{L}) \times \prod_{\dagger} \mathcal{O}(\mathcal{K}_{\dagger}))
\end{array}$$

2-commute. More precisely, there exist natural isomorphisms

$$\phi_{K \rightarrow L} : r_\pi \circ \Psi_K \xrightarrow{\sim} \Psi_L \circ \Psi_{K \rightarrow L}.$$

These isomorphisms satisfy a natural cocycle condition (associated with pairs of epimorphisms  $K \rightarrow L \rightarrow M$ ).

11.11. Applying the functors  $\Psi_K$  componentwise, we get functors

$$\Psi_K : {}^K\mathcal{FS} \longrightarrow \mathcal{L} \rfloor f \dagger f(\mathcal{O}(\mathcal{K}); \mathcal{FS});$$

taking composition with the gluing functor  $g_K$ , (63), we get functors

$$\otimes_K : \mathcal{FS}^{\mathcal{K}} \longrightarrow \mathcal{L} \rfloor f \dagger f(\mathcal{O}(\mathcal{K}); \mathcal{FS}) \quad (68)$$

It follows from the previous lemma that these functors define a braided tensor structure on  $\mathcal{FS}$ .

11.12. Let us define a unit object in  $\mathcal{FS}$  as  $\mathbf{1}_{\mathcal{FS}} = \mathcal{L}(\iota)$  (cf. 6.3). One can show that it is a unit for the braided tensor structure defined above.



## CHAPTER 3. Functor $\Phi$

### 12. FUNCTOR $\Phi$

12.1. Recall the category  $\mathcal{C}$  defined in II.11.3.2 and II.12.2.

Our main goal in this section will be the construction of a tensor functor  $\Phi : \mathcal{FS} \longrightarrow \mathcal{C}$ .

12.2. Recall that we have already defined in 4.5 a functor

$$\Phi : \mathcal{FS} \longrightarrow \mathcal{V}] \sqcup_{\mathfrak{f}}^{\mathcal{X}}.$$

Now we will construct natural transformations

$$\epsilon_i : \Phi_{\lambda}(\mathcal{X}) \longrightarrow \Phi_{\lambda+\gamma'}(\mathcal{X})$$

and

$$\theta_i : \Phi_{\lambda+i'}(\mathcal{X}) \longrightarrow \Phi_{\lambda}(\mathcal{X}).$$

We may, and will, assume that  $\mathcal{X} \in \mathcal{FS}_{\mathfrak{J}}$  for some  $c$ . If  $\lambda \notin X_c$  then there is nothing to do.

Suppose that  $\lambda \in X_c$ ; pick  $\alpha \in \mathbb{N}[\mathbb{I}]$  such that  $\lambda + \alpha' \geq \lambda(\mathcal{X})$ . By definition.

$$\Phi_{\lambda}(\mathcal{X}) = \Phi_{\alpha}(\mathcal{X}_{\lambda+\alpha'}^{\alpha})$$

where  $\Phi_{\alpha}$  is defined in II.7.14 (the definition will be recalled below).

12.3. Pick an unfolding  $\pi : J \longrightarrow I$  of  $\alpha$ , II.6.12; we will use the same notation for the canonical projection

$$\pi : {}^{\pi}\mathbb{A} \longrightarrow \mathcal{A}_{\lambda+\alpha'}^{\alpha} = \mathcal{A}_{\alpha}.$$

Let  $N$  be the dimension of  $\mathcal{A}_{\alpha}$ .

12.4. Recall some notations from II.8.4. For each  $r \in [0, N]$  we have denoted by  $\mathcal{P}_{\nabla}(\mathcal{J}; \infty)$  the set of all maps

$$\varrho : J \longrightarrow [0, r]$$

such that  $\varrho(J)$  contains  $[r]$ . Let us assign to such  $\varrho$  the real point  $w_{\varrho} = (\varrho(j))_{j \in J} \in {}^{\pi}\mathbb{A}$ .

There exists a unique positive facet of  $\mathcal{S}_{\mathbb{R}}$ ,  $F_{\varrho}$  containing  $w_{\varrho}$ . This establishes a bijection between  $\mathcal{P}_{\nabla}(\mathcal{J}; \infty)$  and the set  $\mathcal{F}\text{-}\downarrow_{\nabla}$  of  $r$ -dimensional positive facets. At the same time we have fixed on each  $F_{\varrho}$  a point  $w_{\varrho}$ . This defines cells  $D_{\varrho}^{+} := D_{F_{\varrho}}^{+}$ ,  $S_{\varrho}^{+} := S_{F_{\varrho}}^{+}$ , cf. II.7.2.

Note that this "marking" of positive facets is  $\Sigma_{\pi}$ -invariant. In particular, the group  $\Sigma_{\pi}$  permutes the above mentioned cells.

We will denote by  $\{0\}$  the unique zero-dimensional facet.

12.5. Given a complex  $\mathcal{K}$  from the bounded derived category  $\mathcal{D}^l(\mathcal{A}_\alpha)$ , its inverse image  $\pi^*\mathcal{K}$  is correctly defined as an element of the *equivariant* derived category  $\mathcal{D}^l({}^\pi\mathbb{A}, \Sigma_\pi)$  obtained by localizing the category of  $\Sigma_\pi$ -equivariant complexes on  ${}^\pi\mathbb{A}$ . The direct image  $\pi_*$  acts between equivariant derived categories

$$\pi_* : \mathcal{D}^l({}^\pi\mathbb{A}, \Sigma_\pi) \longrightarrow \mathcal{D}^l(\mathcal{A}_\alpha, \Sigma_\pi)$$

(the action of  $\Sigma_\pi$  on  $\mathcal{A}_\pi$  being trivial).

We have the functor of  $\Sigma_\pi$ -invariants

$$(\bullet)^{\Sigma_\pi} : \mathcal{D}^l(\mathcal{A}_\alpha, \Sigma_\pi) \longrightarrow \mathcal{D}^l(\mathcal{A}_\alpha) \quad (69)$$

12.5.1. **Lemma.** *For every  $\mathcal{K} \in \mathcal{D}^l(\mathcal{A}_\alpha)$  the canonical morphism*

$$\mathcal{K} \longrightarrow (\pi_*\pi^*\mathcal{K})^{\Sigma_\pi}$$

*is an isomorphism.*

**Proof.** The claim may be checked fiberwise. Taking of a fiber commutes with taking  $\Sigma_\pi$ -invariants since our group  $\Sigma_\pi$  is finite and we are living over the field of characteristic zero, hence  $(\bullet)^{\Sigma_\pi}$  is exact. After that, the claim follows from definitions.  $\square$

12.5.2. **Corollary.** *For every  $\mathcal{K} \in \mathcal{D}^l(\mathcal{A}_\alpha)$  we have canonical isomorphism*

$$R\Gamma(\mathcal{A}_\alpha; \mathcal{K}) \xrightarrow{\sim} \mathcal{R}\Gamma({}^\pi\mathbb{A}; \pi^*\mathcal{K})^{\Sigma_\pi} \quad \square \quad (70)$$

12.6. Following II.7.13, consider the sum of coordinates function

$$\sum t_j : {}^\pi\mathbb{A} \longrightarrow \mathbb{A}^k;$$

and for  $\mathcal{L} \in \mathcal{D}({}^\pi\mathbb{A}; \mathcal{S})$  let  $\Phi_{\sum t_j}(\mathcal{L})$  denote the fiber at the origin of the corresponding vanishing cycles functor. If  $H \subset {}^\pi\mathbb{A}$  denotes the inverse image  $(\sum t_j)^{-1}(\{1\})$  then we have canonical isomorphisms

$$\Phi_{\sum t_j}(\mathcal{L}) \cong \mathcal{R}\Gamma({}^\pi\mathbb{A}, \mathbb{H}; \mathcal{L}) \cong \Phi_{\{1\}}^+(\mathcal{L}) \quad (71)$$

The first one follows from the definition of vanishing cycles and the second one from homotopy argument.

Note that the if  $\mathcal{L} = \pi^*\mathcal{K}$  for some  $\mathcal{K} \in \mathcal{D}(\mathcal{A}_\alpha; \mathcal{S})$  then the group  $\Sigma_\pi$  operates canonically on all terms of (71), and the isomorphisms are  $\Sigma_\pi$ -equivariant.

Let us use the same notation

$$\sum t_j : \mathcal{A}_\alpha \longrightarrow \mathbb{A}^k$$

for the descended function, and for  $\mathcal{K} \in \mathcal{D}(\mathcal{A}_\alpha; \mathcal{S})$  let  $\Phi_{\sum t_j}(\mathcal{K})$  denote the fiber at the origin of the corresponding vanishing cycles functor. If  $\mathcal{K}$  belongs to  $\mathcal{M}(\mathcal{A}_\alpha; \mathcal{S})$  then  $\Phi_{\sum t_j}(\mathcal{K})$  reduces to a single vector space and this is what we call  $\Phi_\alpha(\mathcal{K})$ .

If  $\bar{H}$  denotes  $\pi(H) = (\sum t_j)^{-1}(\{1\}) \subset \mathcal{A}_\alpha$  then we have canonical isomorphism

$$\Phi_{\sum t_j}(\mathcal{K}) \cong \mathcal{R}\Gamma(\mathcal{A}_\alpha, \bar{\mathcal{H}}; \mathcal{K})$$

**12.7. Corollary.** (i) *For every  $\mathcal{K} \in \mathcal{D}(\mathcal{A}_\alpha; \mathcal{S})$  we have a canonical isomorphism*

$$i_\pi : \Phi_{\sum t_j}(\mathcal{K}) \xrightarrow{\sim} \Phi_{\{I\}}^+(\pi^*\mathcal{K})^{\Sigma_\pi}. \quad (72)$$

(ii) *This isomorphism does not depend on the choice of an unfolding  $\pi : J \longrightarrow I$ .*

Let us explain what (ii) means. Suppose  $\pi' : J' \longrightarrow I$  be another unfolding of  $\alpha$ . There exists (a non unique) isomorphism

$$\rho : J \xrightarrow{\sim} J'$$

such that  $\pi' \circ \rho = \pi$ . It induces isomorphisms

$$\rho^* : \Sigma_{\pi'} \xrightarrow{\sim} \Sigma_\pi$$

(conjugation by  $\rho$ ), and

$$\rho^* : \Phi_{\{0\}}^+((\pi')^*\mathcal{K}) \xrightarrow{\sim} \Phi_{\{I\}}^+(\pi^*\mathcal{K})$$

such that

$$\rho^*(\sigma x) = \rho^*(\sigma)\rho^*(x),$$

$\sigma \in \Sigma_{\pi'}$ ,  $x \in \Phi_{\{0\}}^+((\pi')^*\mathcal{K})$ . Passing to invariants, we get an isomorphism

$$\rho^* : \Phi_{\{0\}}^+((\pi')^*\mathcal{K})^{\Sigma_{\pi'}} \xrightarrow{\sim} \Phi_{\{I\}}^+(\pi^*\mathcal{K})^{\Sigma_\pi}.$$

Now (ii) means that  $i_\pi \circ \rho^* = i_{\pi'}$ . As a consequence, the last isomorphism does not depend on the choice of  $\rho$ .

**Proof.** Part (i) follows from the preceding discussion and 12.5.2.

As for (ii), it suffices to prove that any automorphism  $\rho : J \xrightarrow{\sim} J$  respecting  $\pi$  induces the identity automorphism of the space of invariants  $\Phi_{\{0\}}^+(\pi^*\mathcal{K})^{\Sigma_\pi}$ . But the action of  $\rho$  on the space  $\Phi_{\{0\}}^+(\pi^*\mathcal{K})$  comes from the action of  $\Sigma_\pi$  on this space, and our claim is obvious.  $\square$

In computations the right hand side of (72) will be used as a *de facto* definition of  $\Phi_\alpha$ .

**12.8.** Suppose that  $\alpha = \sum a_i i$ ; pick an  $i$  such that  $a_i > 0$ .

Let us introduce the following notation. For a partition  $J = J_1 \amalg J_2$  and a positive  $d$  let  $\mathbb{A}^{\mathbb{J}_1, \mathbb{J}_2}()$  denote an open subspace of  ${}^\pi\mathbb{A}$  consisting of all points  $\mathbf{t} = (t_j)$  such that  $|t_j| > d$  (resp.,  $|t_j| < d$ ) if  $j \in J_1$  (resp.,  $j \in J_2$ ).

We have obviously

$$\pi^{-1}(\mathcal{A}^{\lambda, \alpha-}()) = \coprod_{|\in \pi^{-\infty}()} \mathbb{A}^{\{\mathbb{J}\}, \mathbb{J}-\{\mathbb{J}\}}() \quad (73)$$

12.9. For  $j \in \pi^{-1}(i)$  let  $\pi_j : J - \{j\} \longrightarrow I$  denote the restriction of  $\pi$ ; it is an unfolding of  $\alpha - i$ . The group  $\Sigma_{\pi_j}$  may be identified with the subgroup of  $\Sigma_\pi$  consisting of automorphisms stabilizing  $j$ . For  $j', j'' \in J$  let  $(j'j'')$  denotes their transposition. We have

$$\Sigma_{\pi_{j''}} = (j'j'')\Sigma_{\pi_{j'}}(j'j'')^{-1} \quad (74)$$

For a fixed  $j \in \pi^{-1}(i)$  we have a partition into cosets

$$\Sigma_\pi = \coprod_{j' \in \pi^{-1}(i)} \Sigma_{\pi_j}(jj') \quad (75)$$

12.10. For  $j \in J$  let  $F_j$  denote a one-dimensional facet corresponding to the map  $\varrho_j : J \longrightarrow [0, 1]$  sending all elements to 0 except for  $j$  being sent to 1 (cf. 12.4).

For  $\mathcal{K} \in \mathcal{D}(\mathcal{A}_\alpha; \mathcal{S})$  we have canonical and variation maps

$$v_j : \Phi_{\{0\}}^+(\pi^*\mathcal{K}) \xleftarrow{\quad} \Phi_{\mathcal{F}_1}^+(\pi^*\mathcal{K}) : \sqcap_1$$

defined in II, (89), (90). Taking their sum, we get maps

$$v_i : \Phi_{\{0\}}^+(\pi^*\mathcal{K}) \xleftarrow{\quad} \oplus_{|\in \pi^{-\infty}(\cdot)} \Phi_{\mathcal{F}_1}^+(\pi^*\mathcal{K}) : \sqcap_1 \quad (76)$$

Note that the group  $\Sigma_\pi$  operates naturally on both spaces and both maps  $v_i$  and  $u_i$  respect this action.

Let us consider more attentively the action of  $\Sigma_\pi$  on  $\oplus_{j \in \pi^{-1}(i)} \Phi_{F_j}^+(\pi^*\mathcal{K})$ . A subgroup  $\Sigma_{\pi_j}$  respects the subspace  $\Phi_{F_j}^+(\pi^*\mathcal{K})$ . A transposition  $(j'j'')$  maps  $\Phi_{F_{j'}}^+(\pi^*\mathcal{K})$  isomorphically onto  $\Phi_{F_{j''}}^+(\pi^*\mathcal{K})$ .

Let us consider the space of invariants

$$(\oplus_{j \in \pi^{-1}(i)} \Phi_{F_j}^+(\pi^*\mathcal{K}))^{\Sigma_\pi}$$

For every  $k \in \pi^{-1}(i)$  the obvious projection induces isomorphism

$$(\oplus_{j \in \pi^{-1}(i)} \Phi_{F_j}^+(\pi^*\mathcal{K}))^{\Sigma_\pi} \xrightarrow{\sim} (\Phi_{\mathcal{F}_\parallel}^+(\pi^*\mathcal{K}))^{\Sigma_{\pi_\parallel}};$$

therefore for two different  $k, k' \in \pi^{-1}(i)$  we get an isomorphism

$$i_{kk'} : (\Phi_{F_k}^+(\pi^*\mathcal{K}))^{\Sigma_{\pi_\parallel}} \xrightarrow{\sim} (\Phi_{\mathcal{F}_\parallel'}^+(\pi^*\mathcal{K}))^{\Sigma_{\pi_\parallel'}} \quad (77)$$

Obviously, it is induced by transposition  $(kk')$ .

12.11. Let us return to the situation 12.2 and apply the preceding discussion to  $\mathcal{K} = \mathcal{X}_{\lambda+\alpha'}^\alpha$ . We have by definition

$$\Phi_\lambda(\mathcal{X}) = \Phi_{\sum \cup_1}(\mathcal{X}_{\lambda+\alpha'}^\alpha) \cong \Phi_{\{i\}}^+(\pi^*\mathcal{X}_{\lambda+\alpha'}^\alpha)^{\Sigma_\pi} \quad (78)$$

On the other hand, let us pick some  $k \in \pi^{-1}(i)$  and a real  $d$ ,  $0 < d < 1$ . The subspace

$$F_k^\perp(d) \subset \mathbb{A}^{\{\mathfrak{J}, \mathbb{J}-\{\mathfrak{J}\}\}} \quad (79)$$

consisting of points  $(t_j)$  with  $t_k = 1$ , is a transversal slice to the face  $F_k$ . Consequently, the factorization isomorphism for  $\mathcal{X}_{\lambda+\alpha'}^\alpha$  lifted to  $\mathbb{A}^{\{\mathfrak{J}\}, \mathbb{J}^{\mathfrak{J}} - \{\mathfrak{J}\}}()$  induces isomorphism

$$\Phi_{F_k}^+(\pi^* \mathcal{X}_{\lambda+\alpha'}^\alpha) \cong \Phi_{\{\mathfrak{I}\}}^+(\pi_{\parallel}^* \mathcal{X}_{\lambda+\alpha'}^{\alpha-\langle \rangle}) \otimes (\mathcal{I}_{\lambda+\langle \rangle}')_{\{\infty\}} = \Phi_{\{\mathfrak{I}\}}^+(\pi_{\parallel}^* \mathcal{X}_{\lambda+\alpha'}^{\alpha-\langle \rangle})$$

Therefore we get isomorphisms

$$\begin{aligned} \Phi_{\lambda+i'}(\mathcal{X}) &= \Phi_{\{\mathfrak{I}\}}^+(\pi_{\parallel}^* \mathcal{X}_{\lambda+\alpha'}^{\alpha-\langle \rangle})^{\Sigma_{\pi_{\parallel}}} \cong \Phi_{\mathcal{F}_{\parallel}}^+(\pi^* \mathcal{X}_{\lambda+\alpha'}^\alpha)^{\Sigma_{\pi_{\parallel}}} \cong \\ &\cong (\oplus_{j \in \pi^{-1}(i)} \Phi_{F_j}^+(\pi^* \mathcal{X}_{\lambda+\alpha'}^\alpha))^{\Sigma_{\pi}} \end{aligned} \quad (80)$$

It follows from the previous discussion that this isomorphism does not depend on the intermediate choice of  $k \in \pi^{-1}(i)$ .

Now we are able to define the operators  $\theta_i, \epsilon_i$ :

$$\epsilon_i : \Phi_{\lambda}(\mathcal{X}) \xleftrightarrow{\quad} \Phi_{\lambda+i'}(\mathcal{X}) : \theta_i$$

By definition, they are induced by the maps  $u_i, v_i$  (cf. (76)) respectively (for  $\mathcal{K} = \mathcal{X}_{\lambda+\alpha'}^\alpha$ ) after passing to  $\Sigma_{\pi}$ -invariants and taking into account isomorphisms (78) and (80).

**12.12. Theorem.** *The operators  $\epsilon_i$  and  $\theta_i$  satisfy the relations II.12.3, i.e. the previous construction defines functor*

$$\tilde{\Phi} : \mathcal{FS} \longrightarrow \tilde{\mathcal{C}}$$

where the category  $\tilde{\mathcal{C}}$  is defined as in loc. cit.

**Proof** will occupy the rest of the section.

**12.13.** Let  $\mathfrak{u}^+$  (resp.,  $\mathfrak{u}^-$ ) denote the subalgebra of  $\mathfrak{u}$  generated by all  $\epsilon_i$  (resp.,  $\theta_i$ ). For  $\beta \in \mathbb{N}[\mathbb{I}]$  let  $\mathfrak{u}_{\beta}^{\pm} \subset \mathfrak{u}^{\pm}$  denote the corresponding homogeneous component.

The proof will go as follows. First, relations II.12.3 (z), (a), (b) are obvious. We will do the rest in three steps.

*Step 1.* Check of (d). This is equivalent to showing that the action of operators  $\theta_i$  correctly defines maps

$$\mathfrak{u}_{\beta}^- \otimes \Phi_{\lambda+\beta'}(\mathcal{X}) \longrightarrow \Phi_{\lambda}(\mathcal{X})$$

for all  $\beta \in \mathbb{N}[\mathbb{I}]$ .

*Step 2.* Check of (e). This is equivalent to showing that the action of operators  $\epsilon_i$  correctly defines maps

$$\mathfrak{u}_{\beta}^+ \otimes \Phi_{\lambda}(\mathcal{X}) \longrightarrow \Phi_{\lambda+\beta'}(\mathcal{X})$$

for all  $\beta \in \mathbb{N}[\mathbb{I}]$ .

*Step 3.* Check of (c).

12.14. Let us pick an arbitrary  $\beta = \sum b_i i \in \mathbb{N}[\mathbb{I}]$  and  $\alpha \in \mathbb{N}[\mathbb{I}]$  such that  $\lambda + \alpha' \geq \lambda(\mathcal{X})$  and  $\alpha \geq \beta$ . We pick the data from 12.3. In what follows we will generalize the considerations of 12.8 — 12.11.

Let  $U(\beta)$  denote the set of all subsets  $J' \subset J$  such that  $|J' \cap \pi^{-1}(i)| = b_i$  for all  $i$ . Thus, for such  $J'$ ,  $\pi_{J'} := \pi|_{J'} : J' \longrightarrow I$  is an unfolding of  $\beta$  and  $\pi_{J-J'}$  — an unfolding of  $\alpha - \beta$ . We have a disjoint sum decomposition

$$\pi^{-1}(\mathcal{A}^{\beta, \alpha - \beta}(\square)) = \coprod_{J' \in U(\beta)} \mathbb{A}^{\mathbb{J}', \mathbb{J} - \mathbb{J}'}() \quad (81)$$

(cf. (73)).

12.15. For  $J' \in U(\beta)$  let  $F_{J'}$  denote a one-dimensional facet corresponding to the map  $\varrho_{J'} : J \longrightarrow [0, 1]$  sending  $j \notin J'$  to 0  $j \in J'$  — to 1 (cf. 12.4).

For  $\mathcal{K} \in \mathcal{D}(\mathcal{A}_\alpha; \mathcal{S})$  we have canonical and variation maps

$$v_{J'} : \Phi_{\{0\}}^+(\pi^* \mathcal{K}) \xleftarrow{\quad} \Phi_{\mathcal{F}_{J'}}^+(\pi^* \mathcal{K}) : \square_{J'}$$

Taking their sum, we get maps

$$v_\beta : \Phi_{\{0\}}^+(\pi^* \mathcal{K}) \xleftarrow{\quad} \oplus_{J' \in U(\beta)} \Phi_{\mathcal{F}_{J'}}^+(\pi^* \mathcal{K}) : \square_\beta \quad (82)$$

The group  $\Sigma_\pi$  operates naturally on both spaces and both maps  $v_\beta$  and  $u_\beta$  respect this action.

A subgroup  $\Sigma_{J'}$  respects the subspace  $\Phi_{F_{J'}}^+(\pi^* \mathcal{K})$ . The projection induces an isomorphism

$$(\oplus_{J' \in U(\beta)} \Phi_{F_{J'}}^+(\pi^* \mathcal{K}))^{\Sigma_\pi} \xrightarrow{\sim} \Phi_{\mathcal{F}_{J'}}^+(\pi^* \mathcal{K})^{\Sigma_{J'}}.$$

We have the crucial

12.16. **Lemma.** *Factorization isomorphism for  $\mathcal{X}$  induces canonical isomorphism*

$$\mathbf{u}_\beta^- \otimes \Phi_{\lambda + \beta'}(\mathcal{X}) \cong (\oplus_{J' \in U(\beta)} \Phi_{\mathcal{F}_{J'}}^+(\pi^* \mathcal{X}_{\lambda + \alpha'}^\alpha))^{\Sigma_\pi} \quad (83)$$

**Proof.** The argument is the same as in 12.11, using II, Thm. 6.16.  $\square$

12.17. As a consequence, passing to  $\Sigma_\pi$ -invariants in (82) (for  $\mathcal{K} = \mathcal{X}_{\lambda + \alpha'}^\alpha$ ) we get the maps

$$\epsilon_\beta : \Phi_\lambda(\mathcal{X}) \xleftarrow{\quad} \mathbf{u}_\beta^- \otimes \Phi_{\lambda + \beta'}(\mathcal{X}) : \theta_\beta$$

12.18. **Lemma.** *The maps  $\theta_\beta$  provide  $\Phi(\mathcal{X})$  with a structure of a left module over the negative subalgebra  $\mathfrak{u}^-$ .*

**Proof.** We must prove the associativity. It follows from the associativity of factorization isomorphisms.  $\square$

This lemma proves relations II.12.3 (d) for operators  $\theta_i$ , completing Step 1 of the proof of our theorem.

12.19. Now let us consider operators

$$\epsilon_\beta : \Phi_\lambda(\mathcal{X}) \longrightarrow \mathfrak{u}_\beta^- \otimes \Phi_{\lambda+\beta'}(\mathcal{X}).$$

By adjointness, they induce operators

$$\mathfrak{u}_\beta^{-*} \otimes \Phi_\lambda(\mathcal{X}) \longrightarrow \Phi_{\lambda+\beta'}(\mathcal{X})$$

The bilinear form  $S$ , II.2.10, induces isomorphisms

$$S : \mathfrak{u}_\beta^{-*} \xrightarrow{\sim} \mathfrak{u}_\beta^-;$$

let us take their composition with the isomorphism of algebras

$$\mathfrak{u}^- \xrightarrow{\sim} \mathfrak{u}^+$$

sending  $\theta_i$  to  $\epsilon_i$ . We get isomorphisms

$$S' : \mathfrak{u}_\beta^{-*} \xrightarrow{\sim} \mathfrak{u}_\beta^+$$

Using  $S'$ , we get from  $\epsilon_\beta$  operators

$$\mathfrak{u}_\beta^+ \otimes \Phi_\lambda(\mathcal{X}) \longrightarrow \Phi_{\lambda+\beta'}(\mathcal{X})$$

12.19.1. **Lemma.** *The above operators provide  $\Phi(\mathcal{X})$  with a structure of a left  $\mathfrak{u}^+$ -module.*

*For  $\beta = i$  they coincide with operators  $\epsilon_i$  defined above.*

This lemma completes Step 2, proving relations II.12.3 (e) for generators  $\epsilon_i$ .

12.20. Now we will perform the last Step 3 of the proof, i.e. prove the relations II.12.3 (c) between operators  $\epsilon_i, \theta_j$ . Consider a square

$$\begin{array}{ccccc}
& & \Phi_{\lambda+i'}(\mathcal{X}) & & \\
& \nearrow \epsilon_i & & \searrow \theta_j & \\
\Phi_{\lambda}(\mathcal{X}) & & & & \Phi_{\lambda-j'+i'}(\mathcal{X}) \\
& \searrow \theta_j & & \nearrow \epsilon_i & \\
& & \Phi_{\lambda-j'}(\mathcal{X}) & & 
\end{array}$$

We have to prove that

$$\epsilon_i \theta_j - \zeta^{i \cdot j} \theta_j \epsilon_i = \delta_{ij} (1 - \zeta^{-2\lambda \cdot i'}) \quad (84)$$

(cf. (2)).

As before, we may and will suppose that  $\mathcal{X} \in \mathcal{FS}_{\mathbb{J}}$  and  $\lambda \in X_c$  for some  $c$ . Choose  $\alpha \in \mathbb{N}[\mathbb{I}]$  such that  $\alpha \geq i$ ,  $\alpha \geq j$  and  $\lambda - j' + \alpha' \geq \lambda(\mathcal{X})$ . The above square may be identified with the square

$$\begin{array}{ccccc}
& & \Phi_{\alpha-i-j}(\mathcal{X}_{\lambda-|'+\alpha'}^{\alpha-|-|}) & & \\
& \nearrow \epsilon_i & & \searrow \theta_j & \\
\Phi_{\alpha-j}(\mathcal{X}_{\lambda-|'+\alpha'}^{\alpha-|-|}) & & & & \Phi_{\alpha-i}(\mathcal{X}_{\lambda-|'+\alpha'}^{\alpha-|}) \\
& \searrow \theta_j & & \nearrow \epsilon_i & \\
& & \Phi_{\alpha}(\mathcal{X}_{\lambda-|'+\alpha'}^{\alpha}) & & 
\end{array}$$

12.21. Choose an unfolding  $\pi : J \longrightarrow I$  of  $\alpha$ ; let  $\Sigma = \Sigma_{\pi}$  be its automorphism group, and  $\pi : \mathbb{A} = {}^{\pi}\mathbb{A} \longrightarrow \mathcal{A}_{\alpha}$  denote the corresponding covering. We will reduce our proof to certain statements about (vanishing cycles of) sheaves on  $\mathbb{A}$ .

Let us introduce a vector space

$$V = H^0 \Phi_{\{0\}}^+ (\pi^* \mathcal{X}_{\lambda-|'+\alpha'}^{\alpha});$$

the group  $\Sigma$  operates on it, and we have

$$\Phi_{\alpha}(\mathcal{X}_{\lambda-|'+\alpha'}^{\alpha}) \cong \mathcal{V}^{\Sigma} \quad (85)$$

For each  $k \in J$  we have a positive one-dimensional facet  $F_k \subset \mathbb{A}_{\mathbb{R}}$  defined as in 12.10. Denote

$$V_k = H^0 \Phi_{F_k}^+ (\pi^* \mathcal{X}_{\lambda-|'+\alpha'}^{\alpha});$$



we have canonically

$$\Phi_{\alpha-p}(\mathcal{X}_{\lambda-|\prime'+\alpha'}^{\alpha-}) \cong (\oplus_{\|\in\pi^{-\infty}(\prime)} \mathcal{V}_{\|\})^{\Sigma} \quad (86)$$

for each  $p \in I$ ,  $p \leq \alpha$ , cf. 12.11.

12.21.1. We have to extend considerations of 12.11 to two-dimensional facets. For each pair of different  $k, l \in J$  such that  $\pi(k) = i, \pi(l) = j$ , let  $F_{kl}$  denote a two-dimensional positive facet corresponding to the map  $\varrho : J \longrightarrow [0, 2]$  sending  $k$  to 1,  $l$  — to 2 and all other elements — to zero (cf. 12.4). Set

$$V_{kl} = H^0 \Phi_{F_{kl}}^+(\pi^* \mathcal{X}_{\lambda-|\prime'+\alpha'}^{\alpha}).$$

Again, due to equivariance of our sheaf, the group  $\Sigma$  operates on  $\oplus V_{kl}$  in such a way that

$$\sigma(V_{kl}) = V_{\sigma(k)\sigma(l)}.$$

Let

$$\pi_{kl} : J - \{k, l\} \longrightarrow I$$

be the restriction of  $\pi$ . It is an unfolding of  $\alpha - i - j$ ; let  $\Sigma_{kl}$  denote its automorphism group. Pick  $d_1, d_2$  such that  $0 < d_2 < 1 < d_1 < 2$ . The subspace

$$F_{kl}^{\perp} \subset \mathbb{A}^{\{\leq, \{\top\}, \mathbb{J} - \{\top, \leq\}\}(\mu, \neq)} \quad (87)$$

consisting of all points  $(t_j)$  such that  $t_k = 1$  and  $t_l = 2$  is a transversal slice to  $f_{kl}$ . Consequently, factorization axiom implies canonical isomorphism

$$\Phi_{F_{kl}}^+(\pi^* \mathcal{X}_{\lambda-|\prime'+\alpha'}^{\alpha}) \cong \Phi_{\{l\}}^+(\pi_{\|\uparrow}^* \mathcal{X}_{\lambda-|\prime'+\alpha'}^{\alpha-}) \otimes (\mathcal{I}_{\lambda+\prime'})_{\{\infty\}} \otimes (\mathcal{I}_{\lambda})_{\{\infty\}} = \Phi_{\{l\}}^+(\pi_{\|\uparrow}^* \mathcal{X}_{\lambda-|\prime'+\alpha'}^{\alpha-})$$

*Symmetry.* Interchanging  $k$  and  $l$ , we get isomorphisms

$$t : V_{kl} \xrightarrow{\sim} V_{lk} \quad (88)$$

Passing to  $\Sigma$ -invariants, we get isomorphisms

$$\begin{aligned} \Phi_{\alpha-i-j}(\mathcal{X}_{\lambda-|\prime'+\alpha'}^{\alpha-}) &= \Phi_{\{l\}}^+(\pi_{\|\uparrow}^* \mathcal{X}_{\lambda-|\prime'+\alpha'}^{\alpha-})^{\Sigma_{\|\uparrow}} \cong \\ &\cong \Phi_{F_{kl}}^+(\pi^* \mathcal{X}_{\lambda-|\prime'+\alpha'}^{\alpha})^{\Sigma_{\|\uparrow}} \cong (\oplus_{\|\in\pi^{-\infty}(\prime)} \oplus_{\|\in\pi^{-\infty}(\prime), \|\neq\uparrow} \mathcal{V}_{\|\uparrow})^{\Sigma} \end{aligned} \quad (89)$$

(cf. (80)).

12.22. The canonical and variation maps induce linear operators

$$v_k : V \xleftarrow{\quad} V_k : u^k$$

and

$$v_{lk}^k : V_k \xleftarrow{\quad} V_{lk} : u_k^{lk}$$

which are  $\Sigma$ -equivariant in the obvious sense. Taking their sum, we get operators

$$V \xleftarrow{\quad} \oplus_{k \in \pi^{-1}(p)} V_k \xleftarrow{\quad} \oplus_{l \in \pi^{-1}(q), k \in \pi^{-1}(p)} V_{lk}$$

which induce, after passing to  $\Sigma$ -invariants, operators  $\epsilon_p, \epsilon_q$  and  $\theta_p, \theta_q$ .

Our square takes a form

$$\begin{array}{ccc}
 (\oplus_{k \in \pi^{-1}(j), l \in \pi^{-1}(i), k \neq l} V_{lk})^\Sigma & \xrightarrow{\sim} & (\oplus_{k \in \pi^{-1}(j), l \in \pi^{-1}(i), k \neq l} V_{kl})^\Sigma \\
 \uparrow \sum v_{lk}^k & & \downarrow \sum u_l^{kl} \\
 (\oplus_{k \in \pi^{-1}(j)} V_k)^\Sigma & & (\oplus_{l \in \pi^{-1}(i)} V_l)^\Sigma \\
 \searrow \sum u^k & & \nearrow \sum v_l \\
 & V^\Sigma &
 \end{array}$$

Now we will formulate two relations between  $u$  and  $v$  which imply the necessary relations between  $\epsilon$  and  $\theta$ .

**12.23. Lemma.** *Suppose that  $i = j$  and  $\pi(k) = i$ . Consider operators*

$$u^k : V_k \longleftrightarrow V : v_k.$$

*The composition  $v_k u^k$  is equal to the multiplication by  $1 - \zeta^{-2\lambda \cdot i'}$ .*

**Proof.** Consider the transversal slice  $F_k^\perp(d)$  to the face  $F_k$  as in 12.11. It follows from the definition of the canonical and variation maps, II.7.10, that composition  $v_k u^k$  is equal to  $1 - T^{-1}$  where  $T$  is the monodromy acquired by  $\Phi_{\{0\}}^+(\pi_k^* \mathcal{X}_{\lambda - \gamma' + \alpha'}^{\alpha -})$  when the point  $t_k$  moves around the disc of radius  $d$  where all other points are living. By factorization,  $T = \zeta^{2\lambda \cdot i'}$ .  $\square$

**12.24. Lemma.** *For  $k \in \pi^{-1}(j)$ ,  $l \in \pi^{-1}(i)$ ,  $k \neq l$ , consider the pentagon*

$$\begin{array}{ccc}
 V_{lk} & \xrightarrow{\sim} & V_{kl} \\
 \uparrow v_{lk}^k & & \downarrow u_l^{kl} \\
 V_k & & V_l \\
 \searrow u^k & & \nearrow v_l \\
 & V &
 \end{array}$$

We have

$$v_l u^k = \zeta^{i \cdot j} u_k^{kl} \circ t \circ v_{lk}^k.$$

This lemma is a consequence of the following more general statement.

12.25. Let  $\mathcal{K} \in \mathcal{D}(\mathcal{A}; \mathcal{S})$  be arbitrary. We have naturally

$$\Phi_{F_{kl}}^+(\mathcal{K}) \cong \Phi_{\{l\}}^+(\mathcal{K}|_{\mathcal{F}_{\parallel\uparrow}^+(\lceil\infty, \lceil\epsilon\rceil[-\epsilon])})$$

Let

$$t : \Phi_{F_{kl}}^+(\mathcal{K}) \xrightarrow{\sim} \Phi_{\mathcal{F}_{\parallel\uparrow}}^+(\mathcal{K})$$

be the monodromy isomorphism induced by the travel of the point  $t_l$  in the upper half plane to the position to the left of  $t_k$  (outside the disk of radius  $d_1$ ).

12.25.1. **Lemma.** *The composition*

$$v_{F_l}^{\{0\}} \circ u_{\{0\}}^{F_k} : \Phi_{F_k}^+(\mathcal{K}) \longrightarrow \Phi_{\mathcal{F}_{\uparrow}}^+(\mathcal{K})$$

is equal to  $u_{F_l}^{F_{kl}} \circ t \circ v_{F_{lk}}^{F_k}$ .

12.26. It remains to note that due to 12.22 the desired relation (84) is a formal consequence of lemmas 12.23 and 12.24. This completes the proof of Theorem 12.12.  $\square$

12.27. Taking composition of  $\tilde{\Phi}$  with an inverse to the equivalence  $Q$ , II (143), we get a functor

$$\Phi : \mathcal{FS} \longrightarrow \mathcal{C} \tag{90}$$

### 13. MAIN PROPERTIES OF $\Phi$

#### TENSOR PRODUCTS

13.1. **Theorem.**  $\Phi$  is a tensor functor, i.e. we have natural isomorphisms

$$\Phi(\mathcal{X} \dot{\otimes} \mathcal{Y}) \xrightarrow{\sim} \Phi(\mathcal{X}) \otimes \Phi(\mathcal{Y})$$

satisfying all necessary compatibilities.

**Proof** follows from the Additivity theorem, II.9.3.  $\square$

#### DUALITY

13.2. Let  $\tilde{\mathcal{C}}_{\zeta^{-1}}$  denote the category  $\tilde{\mathcal{C}}$  with the value of parameter  $\zeta$  changed to  $\zeta^{-1}$ . The notations  $\mathcal{FS}_{\zeta^{-\infty}}$ , etc. will have the similar meaning.

Let us define a functor

$$D : \tilde{\mathcal{C}}^{opp} \longrightarrow \tilde{\mathcal{C}}_{\zeta^{-1}} \tag{91}$$

as follows. By definition, for  $V = \oplus V_\lambda \in Ob \tilde{\mathcal{C}}^{opp} = Ob \tilde{\mathcal{C}}$ ,

$$(DV)_\lambda = V_\lambda^*,$$

and operators

$$\theta_{i,DV} : (DV)_\lambda \xleftarrow{\quad} (DV)_{\lambda-i'} : \epsilon_{i,DV}$$

are defined as

$$\epsilon_{i,DV} = \theta_{i,V}^*; \theta_{i,DV} = -\zeta^{2\lambda \cdot i'} \epsilon_{i,V}^*.$$

On morphisms  $D$  is defined in the obvious way. One checks directly that  $D$  is well defined, respects tensor structures, and is an equivalence.

13.3. Let us define a functor

$$D : \mathcal{FS}^{\vee\vee} \longrightarrow \mathcal{FS}_{\zeta^{-\infty}} \quad (92)$$

as follows. For  $\mathcal{X} \in \mathcal{O}[\mathcal{FS}^{\vee\vee}] = \mathcal{O}[\mathcal{FS}]$  we set  $\lambda(D\mathcal{X}) = \lambda(\mathcal{X})$ ;  $(D\mathcal{X})^\alpha = \mathcal{D}(\mathcal{X}^\alpha)$  where  $D$  in the right hand side is Verdier duality. Factorization isomorphisms for  $D\mathcal{X}$  are induced in the obvious way from factorization isomorphisms for  $\mathcal{X}$ . The value of  $D$  on morphisms is defined in the obvious way.

$D$  is a tensor equivalence.

13.4. **Theorem.** *Functor  $\tilde{\Phi}$  commutes with  $D$ .*

**Proof.** Our claim is a consequence of the following topological remarks.

13.5. Consider a standard affine space  $\mathbb{A} = \mathbb{A}^{\mathbb{J}}$  with a principal stratification  $\mathcal{S}$  as in II.7. Let  $\mathcal{K} \in \mathcal{D}(\mathbb{A}; \mathcal{S})$ , let  $F_j$  be the one-dimensional facet corresponding to an element  $j \in J$  as in 12.10. Consider a transversal slice  $F_j^\perp(d)$  as in 12.11. We have canonically

$$\Phi_{F_j}^+(\mathcal{K}) \cong \Phi_{\{t_j\}}(\mathcal{K}|_{\mathcal{F}_\perp^\perp(\Gamma)}[-\infty]);$$

when the point  $t_j$  moves counterclockwise around the disk of radius  $d$ ,  $\Phi_j^+(\mathcal{K})$  acquires monodromy

$$T_j : \Phi_j^+(\mathcal{K}) \xrightarrow{\sim} \Phi_j^+(\mathcal{K}).$$

13.5.1. **Lemma.** *Consider canonical and variation maps*

$$v_{D\mathcal{K}} : \Phi_{\{0\}}^+(D\mathcal{K}) \xleftarrow{\quad} \Phi_{\mathcal{F}_\perp}^+(\mathcal{D}\mathcal{K}) : \sqcap_{D\mathcal{K}}$$

*Let us identify  $\Phi_{\{0\}}^+(D\mathcal{K}), \Phi_{\mathcal{F}_\perp}^+(\mathcal{D}\mathcal{K})$  with  $\Phi_{\{0\}}^+(\mathcal{K})^*$  and  $\Phi_{F_j}^+(\mathcal{K})^*$  respectively. Then the maps become*

$$v_{D\mathcal{K}} = u_{\mathcal{K}}^*; \quad u_{D\mathcal{K}} = -v_{\mathcal{K}}^* \circ T_j^*$$

The theorem follows from this lemma.  $\square$

STANDARD OBJECTS

13.6. **Theorem.** *We have naturally*

$$\Phi(\mathcal{L}(\Lambda)) \cong \mathcal{L}(\Lambda)$$

for all  $\Lambda \in X$ .

Combining this with Theorem 6.2, we get

13.7. **Theorem.**  $\Phi$  induces bijection between sets of isomorphism classes of irreducibles.

□

13.8. **Verma modules.** Let  $\mathfrak{u}^{\geq 0} \subset \mathfrak{u}$  denote the subalgebra generated by  $\epsilon_i, K_i, K^{-1}$ ,  $i \in I$ . For  $\Lambda \in X$ , let  $\chi_\Lambda$  denote a one-dimensional representation of  $\mathfrak{u}^{\geq 0}$  generated by a vector  $v_\Lambda$ , with the action

$$\epsilon_i v_\Lambda = 0, \quad K_i v_\Lambda = \zeta^{\langle \Lambda, i \rangle} v_\Lambda.$$

Let  $M(\Lambda)$  denote the induced  $\mathfrak{u}$ -module

$$M(\Lambda) = \mathfrak{u} \otimes_{\mathfrak{u}^{\geq 0}} \chi_\Lambda.$$

Equipped with an obvious  $X$ -grading,  $M(\Lambda)$  becomes an object of  $\tilde{\mathcal{C}}$ . We will also use the same notation for the corresponding object of  $\mathcal{C}$ .

13.9. **Theorem.** *The factorizable sheaves  $\mathcal{M}(\Lambda)$  are finite. We have naturally*

$$\Phi(\mathcal{M}(\Lambda)) \cong \mathcal{M}(\Lambda).$$

**Proof** is given in the next two subsections.

13.10. Let us consider the space  $\mathbb{A}$  as in 13.5. Suppose that  $\mathcal{K} \in \mathcal{D}(\mathbb{A}; \mathcal{S})$  has the form  $\mathcal{K} = |_{\mathbb{I}}^* \mathcal{K}$  where

$$j : \mathbb{A} - \bigcup_{\mathbb{J} \in \mathbb{J}} \{\approx_{\mathbb{J}} = \not\approx\} \hookrightarrow \mathbb{A}.$$

Let  $F_\Delta$  be the positive facet whose closure is the main diagonal.

13.10.1. **Lemma.** *The canonical map*

$$u : \Phi_{F_\Delta}^+(\mathcal{K}) \longrightarrow \Phi_{\{\iota\}}^+(\mathcal{K})$$

is an isomorphism.

**Proof.** Pick  $j_0 \in J$ , and consider a subspace  $Y = \{t_{j_0} = 0\} \cup \{t_{j_0} = 1\} \subset \mathbb{A}$ . Set  $\mathcal{K}' = ||_{\mathbb{I}}^* \mathcal{K}$  where

$$k : \mathbb{A} - \bigcup_{\mathbb{J} \in \mathbb{J}} \{\approx_{\mathbb{J}} = \not\approx\} - \bigcup_{\mathbb{J} \in \mathbb{J}} \{\approx_{\mathbb{J}} = \not\approx\} \hookrightarrow \mathbb{A}.$$

We have

$$\Phi_{\{0\}}^+(\mathcal{K}) = \mathcal{R}\Gamma(\mathbb{A}; \mathcal{K}')$$

On the other hand by homotopy we have

$$\Phi_{F_\Delta}^+(\mathcal{K}) \cong \mathcal{R}\Gamma(\{\sqcup_{|,} = \rfloor\}; \mathcal{K}')[-\infty]$$

where  $c$  is any real between 0 and 1. Let us compute  $R\Gamma(\mathbb{A}; \mathcal{K}')$  using the Leray spectral sequence of a projection

$$p : \mathbb{A} \longrightarrow \mathbb{A}^{\mathbb{K}}, (\approx_{\mathbb{J}}) \mapsto \approx_{\mathbb{J}^*}.$$

The complex  $p_*\mathcal{K}'$  is equal to zero at the points  $\{0\}$  and  $\{1\}$ , and is constant with the fiber  $R\Gamma(\{t_{j_0} = c\}; \mathcal{K}')$  over  $c$ . It follows that

$$R\Gamma(\mathbb{A}; \mathcal{K}') \cong \mathcal{R}\Gamma(\mathbb{A}^{\mathbb{K}}; \iota_*\mathcal{K}') \cong \mathcal{R}\Gamma(\{\sqcup_{|,} = \rfloor\}; \mathcal{K}')[-\infty],$$

and the inverse to this isomorphism may be identified with  $u$ .  $\square$

13.11. Suppose we have  $\alpha \in \mathbb{N}[\mathbb{I}]$ , let  $\pi : J \longrightarrow I$ ;  $\pi : \mathbb{A} \longrightarrow \mathcal{A}_\alpha$  be the corresponding unfolding. Let us apply the previous lemma to  $\mathcal{K} = \pi^*\mathcal{M}(\Lambda)^\alpha$ . Note that after passing to  $\Sigma_\pi$ -invariants, the map  $u$  becomes

$$\mathfrak{u}_\alpha^- \longrightarrow \tilde{\Phi}_\alpha(\mathcal{M}(\Lambda))$$

by Theorem II.6.16. This identifies homogeneous components of  $\tilde{\Phi}_\alpha(\mathcal{M}(\Lambda))$  with the components of the Verma module. After that, the action of  $\epsilon_i$  and  $\theta_i$  is identified with the action of  $\mathfrak{u}$  on it. The theorem is proven.  $\square$

## CHAPTER 4. Equivalence.

### 14. TRUNCATION FUNCTORS

14.1. Recall the notations of 5.4. We fix a coset  $X_c \subset X$ , and we denote the subcategory  $\mathcal{FS}_{\leq \lambda; j} \subset \mathcal{FS}$  by  $\mathcal{FS}_{\leq \lambda}$  for simplicity until further notice.

Given  $\lambda \in X$ , we will denote by  $\mathcal{C}_{\leq \lambda} \subset \mathcal{C}$  the full subcategory of all  $\mathfrak{u}$ -modules  $V$  such that  $V_\mu \neq 0$  implies  $\mu \leq \lambda$ . We denote by  $q_\lambda$  the embedding functor  $\mathcal{C}_\lambda \hookrightarrow \mathcal{C}$ . Obviously,  $\Phi(\mathcal{FS}_{\leq \lambda}) \subset \mathcal{C}_\lambda$ .

In this section we will construct functors

$$\sigma_\lambda^!, \sigma_\lambda^* : \mathcal{FS} \longrightarrow \mathcal{FS}_{\leq \lambda}$$

and

$$q_\lambda^!, q_\lambda^* : \mathcal{C} \longrightarrow \mathcal{C}_{\leq \lambda},$$

such that  $\sigma_\lambda^!$  (resp.,  $\sigma_\lambda^*$ ) is right (resp., left) adjoint to  $\sigma_\lambda$  and  $q_\lambda^!$  (resp.,  $q_\lambda^*$ ) is right (resp., left) adjoint to  $q_\lambda$ .

14.2. First we describe  $\sigma_\lambda^*, \sigma_\lambda^!$ . Given a factorizable sheaf  $\mathcal{X} = \{\mathcal{X}^\alpha\}$  with  $\lambda(\mathcal{X}) = \mu \geq \lambda$  we define FS's  $\mathcal{Y} := \sigma_\lambda^* \mathcal{X}$  and  $\mathcal{Z} := \sigma_\lambda^! \mathcal{X}$  as follows.

We set  $\lambda(\mathcal{Y}) = \lambda(\mathcal{Z}) = \lambda$ . For  $\alpha \in \mathbb{N}[\mathbb{I}]$  we set

$$\mathcal{Y}^\alpha = \mathcal{L}' \sigma^* \mathcal{X}^{\alpha + \mu - \lambda}$$

if  $\alpha + \mu - \lambda \in \mathbb{N}[\mathbb{I}]$  and 0 otherwise, and

$$\mathcal{Z}^\alpha = \mathcal{R}' \sigma^! \mathcal{X}^{\alpha + \mu - \lambda}$$

if  $\alpha + \mu - \lambda \in \mathbb{N}[\mathbb{I}]$  and 0 otherwise. Here  $\sigma$  denotes the canonical closed embedding (cf. 2.4)

$$\sigma : \mathcal{A}_\lambda^\alpha \hookrightarrow \mathcal{A}_\mu^{\alpha + \mu - \lambda},$$

and  $L^0 \sigma^*$  (resp.,  $R^0 \sigma^!$ ) denotes the zeroth perverse cohomology of  $\sigma^*$  (resp., of  $\sigma^!$ ).

The factorization isomorphisms for  $\mathcal{Y}$  and  $\mathcal{Z}$  are induced from those for  $\mathcal{X}$ ; associativity is obvious.

14.3. **Lemma.** *Let  $\mathcal{M} \in \mathcal{FS}_{\leq \lambda}$ ,  $\mathcal{X} \in \mathcal{FS}$ . Then*

$$\mathrm{Hom}_{\mathcal{FS}}(\mathcal{X}, \mathcal{M}) = \mathrm{Hom}_{\mathcal{FS}_{\leq \lambda}}(\sigma_\lambda^* \mathcal{X}, \mathcal{M})$$

and

$$\mathrm{Hom}_{\mathcal{FS}}(\mathcal{M}, \mathcal{X}) = \mathrm{Hom}_{\mathcal{FS}_{\leq \lambda}}(\mathcal{M}, \sigma_\lambda^! \mathcal{X})$$

**Proof.** Let  $\mu = \lambda(\mathcal{X})$ . We have

$$\mathrm{Hom}_{\mathcal{A}_\mu^\alpha}(\mathcal{X}_\mu^\alpha, \mathcal{M}_\mu^\alpha) = \mathrm{Hom}_{\mathcal{A}_\mu^{\alpha-\mu+\lambda}}(\sigma_\lambda^* \mathcal{X}_\lambda^{\alpha-\mu+\lambda}, \mathcal{M}_\lambda^{\alpha-\mu+\lambda})$$

by the usual adjointness. Passing to projective limit in  $\alpha$ , we get the desired result for  $\sigma^*$ . The proof for  $\sigma^!$  is similar.  $\square$

14.4. Given  $\lambda \leq \mu \in X_c$ , we denote by  $\sigma_{\lambda \leq \mu}$  the embedding of the full subcategory

$$\sigma_{\lambda \leq \mu} : \mathcal{FS}_{\leq \lambda} \hookrightarrow \mathcal{FS}_{\leq \mu}.$$

Obviously, the functor

$$\sigma_{\lambda \leq \mu}^* := \sigma_\lambda^* \circ \sigma_\mu : \mathcal{FS}_{\leq \mu} \longrightarrow \mathcal{FS}_{\leq \lambda}$$

is left adjoint to  $\sigma_{\lambda \leq \mu}$ . Similarly,  $\sigma_{\lambda \leq \mu}^! := \sigma_\lambda^! \circ \sigma_\mu$  is the right adjoint to  $\sigma_{\lambda \leq \mu}$ .

For  $\lambda \leq \mu \leq \nu$  we have obvious transitivity

$$\sigma_{\lambda \leq \mu}^* \sigma_{\mu \leq \nu}^* = \sigma_{\lambda \leq \nu}^*; \quad \sigma_{\lambda \leq \mu}^! \sigma_{\mu \leq \nu}^! = \sigma_{\lambda \leq \nu}^!.$$

14.5. For each  $\alpha \in \mathbb{N}[\mathbb{I}]$  and  $i \in I$  such that  $\alpha \geq i$  let  $j_{\nu-i' \leq \nu}^\alpha$  denote the open embedding

$$j_{\nu-i' \leq \nu}^\alpha : \mathcal{A}_\nu^\alpha - \sigma(\mathcal{A}_{\nu-}^{\alpha-}) \hookrightarrow \mathcal{A}_\nu^\alpha.$$

Note that the complement of this open subspace is a divisor, so the corresponding extension by zero and by  $*$  functors are  $t$ -exact, cf. [BBD], 4.1.10 (i). Let us define functors

$$j_{\nu-i' \leq \nu!}, j_{\nu-i' \leq \nu*} : \mathcal{FS}_{\leq \nu} \longrightarrow \mathcal{FS}_{\leq \nu}$$

as follows. For a factorizable sheaf  $\mathcal{X} = \{\mathcal{X}_\nu^\alpha\} \in \mathcal{FS}_{\leq \nu}$  we set

$$(j_{\nu-i' \leq \nu!} \mathcal{X})_\nu^\alpha = |\alpha_{\nu-}^\alpha|_{\nu-}^* |_{\nu-}^* \mathcal{X}_\nu^\alpha$$

and

$$(j_{\nu-i' \leq \nu*} \mathcal{X})_\nu^\alpha = |\alpha_{\nu-}^\alpha|_{\nu-}^* |_{\nu-}^* \mathcal{X}_\nu^\alpha,$$

the factorization isomorphisms being induced from those for  $\mathcal{X}$ .

14.6. **Lemma.** *We have natural in  $\mathcal{X} \in \mathcal{FS}_{\leq \nu}$  exact sequences*

$$j_{\nu-i', \nu!} \mathcal{X} \longrightarrow \mathcal{X} \xrightarrow{a'} \sigma_{\nu-}^* \mathcal{X} \longrightarrow \iota$$

and

$$0 \longrightarrow \sigma_{\nu-i', \nu}^! \mathcal{X} \xrightarrow{a'} \mathcal{X} \longrightarrow |_{\nu-}^* \mathcal{X}$$

where the maps  $a$  and  $a'$  are the adjunction morphisms.

**Proof.** Evidently follows from the same claim at each finite level, which is [BBD], 4.1.10 (ii).  $\square$



14.7. Recall (see 13.3) that we have the Verdier duality functor

$$D : \mathcal{FS}^{\vee\vee} \longrightarrow \mathcal{FS}_{\zeta^{-\infty}}.$$

By definition,  $D(\mathcal{FS}_{\leq \lambda}^{\vee\vee}) \subset \mathcal{FS}_{\zeta^{-\infty}; \leq \lambda}$  for all  $\lambda$ .

We have functorial isomorphisms

$$D \circ \sigma_{\lambda}^* \cong \sigma_{\lambda}^! \circ D; \quad D \circ \sigma_{\nu-i', \nu}^* \cong \sigma_{\nu-i', \nu}^! \circ D$$

and

$$D \circ j_{\nu-i', \nu*} \cong j_{\nu-i', \nu!} \circ D$$

After applying  $D$ , one of the exact sequences in 14.6 becomes another one.

14.8. Let us turn to the category  $\mathcal{C}$ . Below we will identify  $\mathcal{C}$  with  $\tilde{\mathcal{C}}$  using the equivalence  $Q$ , cf. II.12.5. In other words, we will regard objects of  $\mathcal{C}$  as  $\mathfrak{u}$ -modules.

For  $\lambda \in X_c$  functors  $q_{\lambda}^!$  and  $q_{\lambda}^*$  are defined as follows. For  $V \in \mathcal{C}$ ,  $q_{\lambda}^! V$  (resp.,  $q_{\lambda}^* V$ ) is the maximal subobject (resp., quotient) of  $V$  belonging to the subcategory  $\mathcal{C}_{\lambda}$ .

For  $\lambda \leq \mu \in X_c$  let  $q_{\lambda \leq \mu}$  denotes an embedding of a full subcategory

$$q_{\lambda \leq \mu} : \mathcal{C}_{\leq \lambda} \hookrightarrow \mathcal{C}_{\leq \mu}$$

Define  $q_{\lambda \leq \mu}^! := q_{\lambda}^! \circ q_{\mu}$ ;  $q_{\lambda \leq \mu}^* := q_{\lambda}^* \circ q_{\mu}$ . Obviously, the first functor is right adjoint, and the second one is left adjoint to  $q_{\lambda \leq \mu}$ . They have an obvious transitivity property.

14.9. Recall that in 13.2 the weight preserving duality equivalence

$$D : \mathcal{C}^{\vee\vee} \longrightarrow \mathcal{C}_{\zeta^{-\infty}}$$

is defined. By definition,  $D(\mathcal{C}_{\leq \lambda}^{\vee\vee}) \subset \mathcal{C}_{\zeta^{-\infty}; \leq \lambda}$  for all  $\lambda$ .

We have functorial isomorphisms

$$D \circ q_{\lambda}^* \cong q_{\lambda}^! \circ D; \quad D \circ q_{\nu-i', \nu}^* \cong q_{\nu-i', \nu}^! \circ D.$$

14.10. Given  $i \in I$ , let us introduce a "Levi" subalgebra  $\mathfrak{l}_i \subset \mathfrak{u}$  generated by  $\theta_j$ ,  $\epsilon_j$ ,  $j \neq i$ , and  $K_i^{\pm}$ . Let  $\mathfrak{p}_i \subset \mathfrak{u}$  denote the "parabolic" subalgebra generated by  $\mathfrak{l}_i$  and  $\epsilon_i$ .

The subalgebra  $\mathfrak{l}_i$  projects isomorphically to  $\mathfrak{p}_i/(\epsilon_i)$  where  $(\epsilon_i)$  is a two-sided ideal generated by  $\epsilon_i$ . Given an  $\mathfrak{l}_i$ -module  $V$ , we can consider it as a  $\mathfrak{p}_i$ -module by restriction of scalars for the projection  $\mathfrak{p}_i \longrightarrow \mathfrak{p}_i/(\epsilon_i) \cong \mathfrak{l}_i$ , and form the induced  $\mathfrak{u}$ -module  $\text{Ind}_{\mathfrak{p}_i}^{\mathfrak{u}} V$  — "generalized Verma".

14.11. Given an  $\mathfrak{u}$ -module  $V \in \mathcal{C}_{\leq \nu}$ , let us consider a subspace

$${}_i V = \bigoplus_{\alpha \in \mathbb{N}[\mathbb{I} - \{\square\}]} V_{\nu - \alpha'} \subset V.$$

It is an  $X$ -graded  $\mathfrak{p}_i$ -submodule of  $V$ . Consequently, we have a canonical element

$$\pi \in \text{Hom}_{\mathcal{C}}(\text{Ind}_{\pi_i}^{\mathfrak{u}} {}_i V, V) = \text{Hom}_{\mathfrak{p}_i}({}_i V, V)$$

corresponding to the embedding  ${}_i V \hookrightarrow V$ .

We will also consider the dual functor

$$V \mapsto D^{-1} \text{Ind}_{\mathfrak{p}_i}^{\mathfrak{u}} (DV).$$

By duality, we have a natural morphism in  $\mathcal{C}$ ,  $V \longrightarrow D^{-1} \text{Ind}_{\mathfrak{p}_i}^{\mathfrak{u}} (DV)$ .

14.12. **Lemma.** *We have natural in  $V \in \mathcal{C}_{\leq \nu}$  exact sequences*

$$\text{Ind}_{\mathfrak{p}_i}^{\mathfrak{u}} {}_i V \xrightarrow{\pi} V \longrightarrow q_{\nu-i' \leq \nu}^* V \longrightarrow 0$$

and

$$0 \longrightarrow q_{\nu-i' \leq \nu}^! V \longrightarrow V \longrightarrow D^{-1} \text{Ind}_{\mathfrak{p}_i}^{\mathfrak{u}} (DV).$$

where the arrows  $V \longrightarrow q_{\nu-i' \leq \nu}^* V$  and  $q_{\nu-i' \leq \nu}^! V \longrightarrow V$  are adjunction morphisms.

**Proof.** Let us show the exactness of the first sequence. By definition,  $q_{\nu-i' \leq \nu}^* V$  is the maximal quotient of  $V$  lying in the subcategory  $\mathcal{C}_{\nu-i' \leq \nu} \subset \mathcal{C}$ . Obviously,  $\text{Coker } \pi \in \mathcal{C}_{\nu-i' \leq \nu}$ . It remains to show that for any morphism  $h : V \longrightarrow W$  with  $W \in \mathcal{C}_{\nu-i' \leq \nu}$ , the composition  $h \circ \pi : \text{Ind}_{\mathfrak{p}_i}^{\mathfrak{u}} {}_i V \longrightarrow W$  is zero. But  $\text{Hom}_{\mathcal{C}}(\text{Ind}_{\mathfrak{p}_i}^{\mathfrak{u}} {}_i V, W) = \text{Hom}_{\mathfrak{p}_i}({}_i V, W) = 0$  by weight reasons.

The second exact sequence is the dual to the first one.  $\square$

14.13. **Lemma.** *We have natural in  $\mathcal{X} \in \mathcal{FS}_{\nu}$  isomorphisms*

$$\Phi j_{\nu-i' \leq \nu!} \mathcal{X} \xrightarrow{\sim} \text{Ind}_{\mathfrak{p}_i}^{\mathfrak{u}} \mathcal{X}(\Phi \mathcal{X})$$

and

$$\Phi j_{\nu-i' \leq \nu*} \mathcal{X} \xrightarrow{\sim} D^{-\infty} \text{Ind}_{\mathfrak{p}_i}^{\mathfrak{u}} \mathcal{X}(D\Phi \mathcal{X})$$

such that the diagram

$$\begin{array}{ccccc} \Phi j_{\nu-i', \nu!} \mathcal{X} & \longrightarrow & \Phi \mathcal{X} & \longrightarrow & \Phi j_{\nu-i', \nu*} \mathcal{X} \\ \downarrow & & \parallel & & \downarrow \\ \text{Ind}_{\mathfrak{p}_i}^{\mathfrak{u}} \mathcal{X}(\Phi \mathcal{X}) & \longrightarrow & \Phi \mathcal{X} & \longrightarrow & D^{-1} \text{Ind}_{\mathfrak{p}_i}^{\mathfrak{u}} \mathcal{X}(D\Phi \mathcal{X}) \end{array}$$

commutes.

14.14. **Lemma.** *Let  $\lambda \in X_c$ . We have natural in  $\mathcal{X} \in \mathcal{FS}$  isomorphisms*

$$\Phi\sigma_\lambda^*\mathcal{X} \cong \Pi_\lambda^*\Phi\mathcal{X}$$

and

$$\Phi\sigma_\lambda^!\mathcal{X} \cong \Pi_\lambda^!\Phi\mathcal{X}$$

**Proof** follows at once from lemmas 14.13, 14.6 and 14.12.  $\square$

## 15. RIGIDITY

15.1. **Lemma.** *Let  $\mathcal{X} \in \mathcal{FS}_{\leq \iota}$ . Then the natural map*

$$a : \mathrm{Hom}_{\mathcal{FS}}(\mathcal{L}(\iota), \mathcal{X}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathcal{L}(\iota), \Phi(\mathcal{X}))$$

*is an isomorphism.*

**Proof.** We know already that  $a$  is injective, so we have to prove its surjectivity. Let  $\mathcal{K}(\iota)$  (resp.,  $K(0)$ ) denote the kernel of the projection  $\mathcal{M}(\iota) \longrightarrow \mathcal{L}(\iota)$  (resp.,  $M(0) \longrightarrow L(0)$ ). Consider a diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}(\mathcal{L}(\iota), \mathcal{X}) & \longrightarrow & \mathrm{Hom}(\mathcal{M}(\iota), \mathcal{X}) & \longrightarrow & \mathrm{Hom}(\mathcal{K}(\iota), \mathcal{X}) \\ & & \downarrow a & & \downarrow b & & \downarrow \\ 0 & \longrightarrow & \mathrm{Hom}(L(0), \Phi(\mathcal{X})) & \longrightarrow & \mathrm{Hom}(M(0), \Phi(\mathcal{X})) & \longrightarrow & \mathrm{Hom}(K(0), \Phi(\mathcal{X})) \end{array}$$

All vertical rows are injective. On the other hand,  $\mathrm{Hom}(M(0), \Phi(\mathcal{X})) = \Phi(\mathcal{X})_{\iota}$ . The last space is isomorphic to a generic stalk of  $\mathcal{X}_\alpha$  for each  $\alpha \in \mathbb{N}[\mathbb{I}]$ , which in turn is isomorphic to  $\mathrm{Hom}_{\mathcal{FS}}(\mathcal{M}(\iota), \mathcal{X})$  by the universal property of the shriek extension. Consequently,  $b$  is isomorphism by the equality of dimensions. By diagram chase, we conclude that  $a$  is isomorphism.  $\square$

15.2. **Lemma.** *For every  $\mathcal{X} \in \mathcal{FS}$  the natural maps*

$$\mathrm{Hom}_{\mathcal{FS}}(\mathcal{L}(\iota), \mathcal{X}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathcal{L}(\iota), \Phi(\mathcal{X}))$$

and

$$\mathrm{Hom}_{\mathcal{FS}}(\mathcal{X}, \mathcal{L}(\iota)) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\Phi(\mathcal{X}), \mathcal{L}(\iota))$$

are isomorphisms.

**Proof.** We have

$$\begin{aligned}
& \mathrm{Hom}_{\mathcal{FS}}(\mathcal{L}(\iota), \mathcal{X}) = \mathrm{Hom}_{\mathcal{FS}}(\mathcal{L}(\iota), \sigma_!^! \mathcal{X}) \\
& \text{(by lemma 14.3)} \\
& = \mathrm{Hom}_{\mathcal{C}}(L(0), \Phi(\sigma_0^! \mathcal{X})) \\
& \text{(by the previous lemma)} \\
& = \mathrm{Hom}_{\mathcal{C}}(L(0), q_0^! \Phi(\mathcal{X})) \\
& \text{(by lemma 14.14)} \\
& = \mathrm{Hom}_{\mathcal{C}}(L(0), \Phi(\mathcal{X})).
\end{aligned}$$

This proves the first isomorphism. The second one follows by duality.  $\square$

15.3. Recall (see [KL]IV, Def. A.5) that an object  $X$  of a tensor category is called *rigid* if there exists another object  $X^*$  together with morphisms

$$i_X : \mathbf{1} \longrightarrow X \otimes X^*$$

and

$$e_X : X^* \otimes X \longrightarrow \mathbf{1}$$

such that the compositions

$$X = \mathbf{1} \otimes X \xrightarrow{i_X \otimes \mathrm{id}} X \otimes X^* \otimes X \xrightarrow{\mathrm{id} \otimes e_X} X$$

and

$$X^* = X^* \otimes \mathbf{1} \xrightarrow{\mathrm{id} \otimes i_X} X^* \otimes X \otimes X^* \xrightarrow{e_X \otimes \mathrm{id}} X^*$$

are equal to  $\mathrm{id}_X$  and  $\mathrm{id}_{X^*}$  respectively. A tensor category is called *rigid* if all its objects are rigid.

15.4. **Theorem.** *All irreducible objects  $\mathcal{L}(\lambda)$ ,  $\lambda \in \mathcal{X}$ , are rigid in  $\mathcal{FS}$ .*

**Proof.** We know (cf. [AJS], 7.3) that  $\mathcal{C}$  is rigid. Moreover, there exists an involution  $\lambda \mapsto \bar{\lambda}$  on  $X$  such that  $L(\lambda)^* = L(\bar{\lambda})$ . Let us define  $\mathcal{L}(\lambda)^* := \mathcal{L}(\bar{\lambda})$ ;  $i_{\mathcal{L}(\lambda)}$  corresponds to  $i_{L(\lambda)}$  under identification

$$\mathrm{Hom}_{\mathcal{FS}}(\mathcal{L}(\iota), \mathcal{L}(\lambda) \otimes \mathcal{L}(\bar{\lambda})) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{L}(\iota), \mathcal{L}(\lambda) \otimes \mathcal{L}(\bar{\lambda}))$$

and  $e_{\mathcal{L}(\lambda)}$  corresponds to  $e_{L(\lambda)}$  under identification

$$\mathrm{Hom}_{\mathcal{FS}}(\mathcal{L}(\bar{\lambda}) \otimes \mathcal{L}(\lambda), \mathcal{L}(\iota)) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{L}(\bar{\lambda}) \otimes \mathcal{L}(\lambda), \mathcal{L}(\iota)),$$

cf. 15.2.  $\square$

## 16. STEINBERG SHEAF

In this section we assume that  $l$  is a positive number prime to 2, 3 and that  $\zeta'$  is a primitive  $(l \cdot \det A)$ -th root of 1 (recall that  $\zeta = (\zeta')^{\det A}$ ).

We fix a weight  $\lambda_0 \in X$  such that  $\langle i, \lambda_0 \rangle = -1 \pmod{l}$  for any  $i \in I$ . Our goal in this section is the proof of the following

**16.1. Theorem.** *The FFS  $\mathcal{L}(\lambda_l)$  is a projective object of the category  $\mathcal{FS}$ .*

**Proof.** We have to check that  $\text{Ext}^1(\mathcal{L}(\lambda_l), \mathcal{X}) = \iota$  for any FFS  $\mathcal{X}$ . By induction on the length of  $\mathcal{X}$  it is enough to prove that  $\text{Ext}^1(\mathcal{L}(\lambda_l), \mathcal{L}) = \iota$  for any simple FFS  $\mathcal{L}$ .

**16.2.** To prove vanishing of  $\text{Ext}^1$  in  $\mathcal{FS}$  we will use the following principle. Suppose  $\text{Ext}^1(\mathcal{X}, \mathcal{Y}) \neq \iota$ , and let

$$0 \longrightarrow \mathcal{Y} \longrightarrow \mathcal{Z} \longrightarrow \mathcal{X} \longrightarrow \iota$$

be the corresponding nonsplit extension. Let us choose a weight  $\lambda$  which is bigger than of  $\lambda(\mathcal{X}), \lambda(\mathcal{Y}), \lambda(\mathcal{Z})$ . Then for any  $\alpha \in \mathbb{N}[\mathbb{I}]$  the sequence

$$0 \longrightarrow \mathcal{Y}_\lambda^\alpha \longrightarrow \mathcal{Z}_\lambda^\alpha \longrightarrow \mathcal{X}_\lambda^\alpha \longrightarrow \iota$$

is also exact, and for  $\alpha \gg 0$  it is also nonsplit (see lemma 5.3). That is, for  $\alpha \gg 0$  we have  $\text{Ext}^1(\mathcal{X}_\lambda^\alpha, \mathcal{Y}_\lambda^\alpha) \neq \iota$  in the category of all perverse sheaves on the space  $\mathcal{A}_\lambda^\alpha$ . This latter Ext can be calculated purely topologically. So its vanishing gives a criterion of  $\text{Ext}^1$ -vanishing in the category  $\mathcal{FS}$ .

**16.3.** In calculating  $\text{Ext}^1(\mathcal{L}(\lambda_l), \mathcal{L}(\mu))$  we will distinguish between the following three cases:

- a)  $\mu \not\geq \lambda_0$ ;
- b)  $\mu = \lambda_0$ ;
- c)  $\mu > \lambda_0$ .

**16.4.** Let us treat the case a).

**16.4.1. Lemma.** *For any  $\alpha \in \mathbb{N}[\mathbb{I}]$  the sheaf  $\mathcal{L}(\lambda_l)_\lambda^\alpha$  is the shriek-extension from the open stratum of toric stratification of  $\mathcal{A}_\lambda^\alpha$ .*

**Proof.** Due to the factorization property it is enough to check that the stalk of  $\mathcal{L}(\lambda_l)_\lambda^\alpha$  at the point  $\{0\} \in \mathcal{A}_\lambda^\alpha$  vanishes for any  $\alpha \in \mathbb{N}[\mathbb{I}], \alpha \neq \emptyset$ . By the Theorem II.8.23, we have  $(\mathcal{L}(\lambda_l)_\lambda^\alpha)_\iota = {}_\alpha C_\mathfrak{f}^\bullet(\mathcal{L}(\lambda_l)) \simeq \iota$  since  $L(\lambda_0)$  is a free  $\mathfrak{f}$ -module by [L] 36.1.5 and Theorem II.11.10(b) and, consequently,  $C_\mathfrak{f}^\bullet(L(\lambda_0)) \simeq H_\mathfrak{f}^0(L(\lambda_0)) = B$  and has weight zero.  $\square$

Returning to the case a), let us choose  $\nu \in X$ ,  $\nu \geq \lambda_0, \nu \geq \mu$ . For any  $\alpha$ , the sheaf  $\mathcal{L}' := \mathcal{L}(\mu)_\nu^\alpha$  is supported on the subspace

$$\mathcal{A}' := \sigma(\mathcal{A}_\mu^{\alpha+\mu-\nu}) \subset \mathcal{A}_\nu^\alpha$$

and  $\mathcal{L}'' := \mathcal{L}(\lambda_r)_\nu^\alpha$  — on the subspace

$$\mathcal{A}'' := \sigma(\mathcal{A}_{\lambda_r}^{\alpha+\lambda_r-\nu}) \subset \mathcal{A}_\nu^\alpha.$$

Let  $i$  denote a closed embedding

$$i : \mathcal{A}'' \hookrightarrow \mathcal{A}_\nu^\alpha$$

and  $j$  an open embedding

$$j : \overset{\circ}{\mathcal{A}}'' := \mathcal{A}'' - \mathcal{A}'' \cap \mathcal{A}' \hookrightarrow \mathcal{A}''.$$

We have by adjointness

$$R\mathrm{Hom}_{\mathcal{A}_\nu^\alpha}(\mathcal{L}'', \mathcal{L}') = \mathcal{R}\mathrm{Hom}_{\mathcal{A}''}(\mathcal{L}'', \mathcal{L}') =$$

(by the previous lemma)

$$= R\mathrm{Hom}_{\overset{\circ}{\mathcal{A}}''}(j^*\mathcal{L}'', |\ast|^!\mathcal{L}') = \iota$$

since obviously  $j^*i^!\mathcal{L}' = \iota$ . This proves the vanishing in the case a).

16.5. In case (b), suppose

$$0 \longrightarrow \mathcal{L}(\lambda_r) \longrightarrow \mathcal{X} \longrightarrow \mathcal{L}(\lambda_r) \longrightarrow \iota$$

is a nonsplit extension in  $\mathcal{FS}$ . Then for  $\alpha \gg 0$  the restriction of  $\mathcal{X}_{\lambda_r}^\alpha$  to the open toric stratum of  $\mathcal{A}_{\lambda_r}^\alpha$  is a nonsplit extension

$$0 \longrightarrow \overset{\bullet}{\mathcal{I}}_{\lambda_0}^\alpha \longrightarrow \overset{\bullet}{\mathcal{X}}_{\lambda_0}^\alpha \longrightarrow \overset{\bullet}{\mathcal{I}}_{\lambda_0}^\alpha \longrightarrow 0$$

(in the category of all perverse sheaves on  $\overset{\bullet}{\mathcal{A}}_{\lambda_0}^\alpha$ ) (we can restrict to the open toric stratum because of lemma 16.4.1). This contradicts to the factorization property of FFS  $\mathcal{X}$ . This contradiction completes the consideration of case (b).

16.6. In case (c), suppose  $\mathrm{Ext}^1(\mathcal{L}(\lambda_r), \mathcal{L}(\mu)) \neq \iota$  whence  $\mathrm{Ext}^1(\mathcal{L}(\lambda_r)_\mu^\alpha, \mathcal{L}(\mu)_\mu^\alpha) \neq \iota$  for some  $\alpha \in \mathbb{N}[\mathbb{I}]$  by the principle 16.2. Here the latter  $\mathrm{Ext}$  is taken in the category of all perverse sheaves on  $\mathcal{A}_\mu^\alpha$ . We have  $\mu - \lambda_0 = \beta'$  for some  $\beta \in \mathbb{N}[\mathbb{I}], \beta \neq \beta'$ .

Let us consider the closed embedding

$$\sigma : \mathcal{A}' := \mathcal{A}_{\lambda_r}^{\alpha-\beta} \hookrightarrow \mathcal{A}_\mu^\alpha;$$

let us denote by  $j$  an embedding of the open toric stratum

$$j : \overset{\bullet}{\mathcal{A}}' := \overset{\bullet}{\mathcal{A}}_{\lambda_0}^{\alpha-\beta} \hookrightarrow \mathcal{A}'.$$

As in the previous case, we have

$$R\mathrm{Hom}_{\mathcal{A}_\mu^\alpha}(\mathcal{L}(\lambda_r)_\mu^\alpha, \mathcal{L}(\mu)_\mu^\alpha) = \mathcal{R}\mathrm{Hom}_{\mathcal{A}'}(\mathcal{L}(\lambda_r)_\mu^\alpha, \sigma^!\mathcal{L}(\mu)_\mu^\alpha) = \mathcal{R}\mathrm{Hom}_{\overset{\bullet}{\mathcal{A}}'}(|\ast|\mathcal{L}(\lambda_r)_\mu^\alpha, |\ast|\sigma^!\mathcal{L}(\mu)_\mu^\alpha).$$

We claim that  $j^* \sigma^! \mathcal{L}(\mu)_\mu^\alpha = \iota$ . Since the sheaf  $\mathcal{L}(\mu)_\mu^\alpha$  is Verdier auto-dual up to replacing  $\zeta$  by  $\zeta^{-1}$ , it suffices to check that  $j^* \sigma^* \mathcal{L}(\mu)_\mu^\alpha = \iota$ .

To prove this vanishing, by factorization property of  $\mathcal{L}(\mu)$ , it is enough to check that the stalk of the sheaf  $\mathcal{L}(\mu)_\mu^\beta$  at the point  $\{0\} \in \mathcal{A}_\mu^\beta$  vanishes.

By the Theorem II.8.23, we have  $(\mathcal{L}(\mu)_\mu^\beta)_! = {}_\beta \mathcal{C}_!^\bullet(\mathcal{L}(\mu))$ . By the Theorem II.11.10 and Shapiro Lemma, we have  ${}_\beta \mathcal{C}_!^\bullet(L(\mu)) \simeq C_U^\bullet(M(\lambda_0) \otimes L(\mu))$ .

By the Theorem 36.1.5. of [L], the canonical projection  $M(\lambda_0) \longrightarrow L(\lambda_0)$  is an isomorphism. By the autoduality of  $L(\lambda_0)$  we have  $C_U^\bullet(L(\lambda_0) \otimes L(\mu)) \simeq R \operatorname{Hom}_U^\bullet(L(\lambda_0), L(\mu)) \simeq 0$  since  $L(\lambda_0)$  is a projective  $U$ -module, and  $\mu \neq \lambda_0$ .

This completes the case c) and the proof of the theorem.  $\square$

## 17. EQUIVALENCE

We keep the assumptions of the previous section.

**17.1. Theorem.** *Functor  $\Phi : \mathcal{FS} \longrightarrow \mathcal{C}$  is an equivalence.*

**17.2. Lemma.** *For any  $\lambda \in X$  the FFS  $\mathcal{L}(\lambda_r) \overset{\bullet}{\otimes} \mathcal{L}(\lambda)$  is projective.*

*As  $\lambda$  ranges through  $X$ , these sheaves form an ample system of projectives in  $\mathcal{FS}$ .*

**Proof.** We have

$$\operatorname{Hom}(\mathcal{L}(\lambda_r) \overset{\bullet}{\otimes} \mathcal{L}(\lambda), ?) = \operatorname{Hom}(\mathcal{L}(\lambda_r), \mathcal{L}(\lambda)^* \overset{\bullet}{\otimes} ?)$$

by the rigidity, and the last functor is exact since  $\overset{\bullet}{\otimes}$  is a biexact functor in  $\mathcal{FS}$ , and  $\mathcal{L}(\lambda_r)$  is projective. Therefore,  $\mathcal{L}(\lambda_r) \overset{\bullet}{\otimes} \mathcal{L}(\lambda)$  is projective.

To prove that these sheaves form an ample system of projectives, it is enough to show that for each  $\mu \in X$  there exists  $\lambda$  such that  $\operatorname{Hom}(\mathcal{L}(\lambda_r) \overset{\bullet}{\otimes} \mathcal{L}(\lambda), \mathcal{L}(\mu)) \neq \iota$ . We have

$$\operatorname{Hom}(\mathcal{L}(\lambda_r) \overset{\bullet}{\otimes} \mathcal{L}(\lambda), \mathcal{L}(\mu)) = \operatorname{Hom}(\mathcal{L}(\lambda), \mathcal{L}(\lambda_r)^* \overset{\bullet}{\otimes} \mathcal{L}(\mu)).$$

Since the sheaves  $\mathcal{L}(\lambda)$  exhaust irreducibles in  $\mathcal{FS}$ , there exists  $\lambda$  such that  $\mathcal{L}(\lambda)$  embeds into  $\mathcal{L}(\lambda_r)^* \overset{\bullet}{\otimes} \mathcal{L}(\mu)$ , hence the last group is non-zero.  $\square$

**17.3. Proof of 17.1.** As  $\lambda$  ranges through  $X$ , the modules  $\Phi(\mathcal{L}(\lambda_r) \overset{\bullet}{\otimes} \mathcal{L}(\lambda)) = \mathcal{L}(\lambda_r) \otimes \mathcal{L}(\lambda)$  form an ample system of projectives in  $\mathcal{C}$ . By the Lemma A.15 of [KL]IV we only have to show that

$$\Phi : \operatorname{Hom}_{\mathcal{FS}}(\mathcal{L}(\lambda_r) \overset{\bullet}{\otimes} \mathcal{L}(\lambda), \mathcal{L}(\lambda_r) \overset{\bullet}{\otimes} \mathcal{L}(\mu)) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(\mathcal{L}(\lambda_r) \otimes \mathcal{L}(\lambda), \mathcal{L}(\lambda_r) \otimes \mathcal{L}(\mu))$$

is an isomorphism for any  $\lambda, \mu \in X$ . We already know that it is an injection. Therefore, it remains to compare the dimensions of the spaces in question. We have

$$\dim \operatorname{Hom}_{\mathcal{FS}}(\mathcal{L}(\lambda_r) \dot{\otimes} \mathcal{L}(\lambda), \mathcal{L}(\lambda_r) \dot{\otimes} \mathcal{L}(\mu)) = \dim \operatorname{Hom}_{\mathcal{FS}}(\mathcal{L}(\lambda_r), \mathcal{L}(\lambda_r) \dot{\otimes} \mathcal{L}(\mu) \dot{\otimes} \mathcal{L}(\lambda)^*)$$

by rigidity,

$$= [\mathcal{L}(\lambda_r) \dot{\otimes} \mathcal{L}(\mu) \dot{\otimes} \mathcal{L}(\lambda)^* : \mathcal{L}(\lambda_r)]$$

because  $\mathcal{L}(\lambda_r)$  is its own indecomposable projective cover in  $\mathcal{FS}$ ,

$$= [L(\lambda_0) \otimes L(\mu) \otimes L(\lambda)^* : L(\lambda_0)]$$

since  $\Phi$  induces an isomorphism of  $K$ -rings of the categories  $\mathcal{FS}$  and  $\mathcal{C}$ ,

$$= \dim \operatorname{Hom}_{\mathcal{C}}(L(\lambda_0) \otimes L(\lambda), L(\lambda_0) \otimes L(\mu))$$

by the same argument applied to  $\mathcal{C}$ . The theorem is proven.  $\square$

## 18. THE CASE OF GENERIC $\zeta$

In this section we suppose that  $\zeta$  is not a root of unity.

18.1. Recall the notations of II.11,12. We have the algebra  $U$  defined in II.12.2, the algebra  $\mathfrak{u}$  defined in II.12.3, and the homomorphism  $R : U \longrightarrow \mathfrak{u}$  defined in II.12.5.

18.1.1. **Lemma.** *The map  $R : U \longrightarrow \mathfrak{u}$  is an isomorphism.*

**Proof** follows from [R], no. 3, Corollaire.  $\square$

18.2. We call  $\Lambda \in X$  dominant if  $\langle i, \Lambda \rangle \geq 0$  for any  $i \in I$ . An irreducible  $U$ -module  $L(\Lambda)$  is finite dimensional if and only if  $\Lambda$  is dominant. Therefore we will need a larger category  $\mathcal{O}$  containing all irreducibles  $L(\Lambda)$ .

Define  $\mathcal{O}$  as a category consisting of all  $X$ -graded  $U$ -modules  $V = \bigoplus_{\mu \in X} V_\mu$  such that

- a)  $V_\mu$  is finite dimensional for any  $\mu \in X$ ;
- b) there exists  $\lambda = \lambda(V)$  such that  $V_\mu = 0$  if  $\mu \not\geq \lambda(V)$ .

18.2.1. **Lemma** *The category  $\mathcal{O}$  is equivalent to the usual category  $\mathcal{O}_{\mathfrak{g}}$  over the classical finite dimensional Lie algebra  $\mathfrak{g}$ .*

**Proof.** See [F].  $\square$

18.3. Let  $W$  denote the Weyl group of our root datum. For  $w \in W$ ,  $\lambda \in X$  let  $w \cdot \lambda$  denote the usual action of  $W$  on  $X$  centered at  $-\rho$ .

Finally, for  $\Lambda \in X$  let  $M(\Lambda) \in \mathcal{O}$  denote the  $U^-$ -free Verma module with highest weight  $\Lambda$ .



18.3.1. **Corollary.** *Let  $\mu, \nu \in X$  be such that  $W \cdot \mu \neq W \cdot \nu$ . Then  $\text{Ext}^\bullet(M(\nu), L(\mu)) = 0$ .*

**Proof.**  $M(\nu)$  and  $L(\mu)$  have different central characters.  $\square$

18.4. **Theorem.** *Functor  $\Phi : \mathcal{FS} \longrightarrow \mathcal{C}$  is an equivalence.*

**Proof.** We know that  $\Phi(\mathcal{L}(\Lambda)) \simeq \mathcal{L}(\Lambda)$  for any  $\Lambda \in X$ . So  $\Phi(\mathcal{X})$  is finite dimensional iff all the irreducible subquotients of  $\mathcal{X}$  are of the form  $\mathcal{L}(\lambda)$ ,  $\lambda$  dominant. By virtue of Lemma 18.2.1 above the category  $\mathcal{C}$  is semisimple: it is equivalent to the category of finite dimensional  $\mathfrak{g}$ -modules. It consists of finite direct sums of modules  $L(\lambda)$ ,  $\lambda$  dominant. So to prove the Theorem it suffices to check semisimplicity of  $\mathcal{FS}$ . Thus the Theorem follows from

18.5. **Lemma.** *Let  $\mu, \nu \in X$  be the dominant weights. Then  $\text{Ext}^1(\mathcal{L}(\mu), \mathcal{L}(\nu)) = \iota$ .*

**Proof.** We will distinguish between the following two cases:

- (a)  $\mu = \nu$ ;
- (b)  $\mu \neq \nu$ .

In calculating  $\text{Ext}^1$  we will use the principle 16.2. The argument in case (a) is absolutely similar to the one in section 16.5, and we leave it to the reader.

In case (b) either  $\mu - \nu \notin Y \subset X$  — and then the sheaves  $\mathcal{L}(\nu)$  and  $\mathcal{L}(\mu)$  are supported on the different connected components of  $\mathcal{A}$ , whence  $\text{Ext}^1$  obviously vanishes, — or there exists  $\lambda \in X$  such that  $\lambda \geq \mu, \nu$ . Let us fix such  $\lambda$ . Suppose  $\text{Ext}^1(\mathcal{L}(\mu), \mathcal{L}(\nu)) \neq \iota$ . Then according to the principle 16.2 there exists  $\alpha \in \mathbb{N}[\mathbb{I}]$  such that  $\text{Ext}^1(\mathcal{L}(\mu)_\lambda^\alpha, \mathcal{L}(\nu)_\lambda^\alpha) \neq \iota$ . The latter Ext is taken in the category of all perverse sheaves on  $\mathcal{A}_\lambda^\alpha$ .

We have  $\text{Ext}^1(\mathcal{L}(\mu)_\lambda^\alpha, \mathcal{L}(\nu)_\lambda^\alpha) = \mathcal{R}^\infty \Gamma(\mathcal{A}_\lambda^\alpha, \mathcal{D}(\mathcal{L}(\mu)_\lambda^\alpha \otimes \mathcal{D}\mathcal{L}(\nu)_\lambda^\alpha))$  where  $D$  stands for Verdier duality, and  $\otimes$  denotes the usual tensor product of constructible complexes.

We will prove that

$$\mathcal{L}(\mu)_\lambda^\alpha \otimes \mathcal{D}\mathcal{L}(\nu)_\lambda^\alpha = \iota \tag{93}$$

and hence we will arrive at the contradiction. Equality (93) is an immediate corollary of the lemma we presently formulate.

For  $\beta \leq \alpha$  let us consider the canonical embedding

$$\sigma : \dot{\mathcal{A}}^{\alpha-\beta} \hookrightarrow \mathcal{A}^\alpha$$

and denote its image by  $\dot{\mathcal{A}}'$  (we omit the lower case indices).

18.5.1. **Lemma.** (i) *If  $\sigma^* \mathcal{L}(\mu)_\lambda^\alpha \neq \iota$  then  $\lambda - \beta \in W \cdot \mu$ .*

(ii) *If  $\sigma^! \mathcal{L}(\mu)_\lambda^\alpha \neq \iota$  then  $\lambda - \beta \in W \cdot \mu$ .*

To deduce Lemma 18.5 from this lemma we notice first that the sheaf  $\mathcal{L}(\mu)_\lambda^\alpha$  is Verdier autodual up to replacing  $\zeta$  by  $\zeta^{-1}$ . Second, since the  $W$ -orbits of  $\nu$  and  $\mu$  are disjoint, we see that over any toric stratum  $\dot{\mathcal{A}} \subset \mathcal{A}^\alpha$  at least one of the factors of (93) vanishes.

It remains to prove Lemma 18.5.1. We will prove (i), while (ii) is just dual. Let us denote  $\beta + \mu - \lambda$  by  $\gamma$ . If  $\gamma \notin \mathbb{N}[\mathbb{I}]$  then (i) is evident. Otherwise, by the factorizability condition it is enough to check that the stalk of  $\mathcal{L}(\mu)_\mu^\gamma$  at the origin in  $\mathcal{A}_\mu^\gamma$  vanishes if  $\mu - \gamma \notin W \cdot \mu$ . Let us denote  $\mu - \gamma$  by  $\chi$ .

By the Theorem II.8.23, we have  $(\mathcal{L}(\mu)_\mu^\gamma)_! = {}_\gamma \mathcal{C}_{\mathcal{U}^-}^\bullet(\mathcal{L}(\mu)) \simeq \mathcal{C}_{\mathcal{U}}^\bullet(\mathcal{M}(\chi) \otimes \mathcal{L}(\mu))$  which is dual to  $\text{Ext}_{\mathcal{U}}^\bullet(M(\chi), L(\mu))$ . But the latter Ext vanishes by the Corollary 18.3.1 since  $W \cdot \chi \neq W \cdot \mu$ .  $\square$

This completes the proof of Lemma 18.5 together with Theorem 18.4.  $\square$

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